# Cournot Competition with Asymmetric Price Caps*

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Price regulation may subject identical goods to different price caps. To analyze the welfare effects of such regulation, I incorporate asymmetric price caps into a quantity competition model. A (non-price) rationing rule determines a firm's inverse residual demand function in the presence of price caps, and the price cap makes the inverse residual demand function flat above the price cap. If only one price cap binds, asymmetric price caps induce a trade-off: When adjusting the binding price cap to increase the total quantity, the production gets more unequal across firms, that is, more inefficient.

**JEL Codes:** D43, L11, L51
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# 1 Introduction

In imperfectly competitive markets, price caps are supposed to improve the welfare by increasing the traded quantity. But if the price cap regulation affects the firms in a duopoly asymmetrically, even if only marginally so, judging price caps by the total quantity alone is shortsighted: The regulation distorts the allocation of production across firms—it causes inefficient production.

Asymmetric price caps arise if a regulator can legally put a price cap on only one firm in a market; either because the state owns the firm or because the firm dominates the market. An example is the German market for postal services: Because of its dominant market position, the former German state monopoly is subject to price caps for the delivery of letters, whereas its competitors are not.[1] Another source of asymmetric price caps are symmetric price caps on differentiated goods: The real price cap is then tighter for firms with better products.[2] Lastly, asymmetric price caps might arise from freezing current prices: Some laws against price gouging restrict price increases of incumbent sellers during crises differently than the prices of entering sellers,[3] and even without such laws prices often fail to adjust upwards.[4]

This chapter is concerned with the questions: How does asymmetric price regulation distort the production in a Cournot duopoly? Can asymmetric price caps still improve the welfare?

To answer these questions, I add asymmetric price caps to the canonical Cournot quantity competition model with increasing marginal cost. The price that the firm anticipates to receive depends on its inverse residual demand and on its price cap. In the canonical model, the inverse residual demand function follows from the prices' rationing the (total) demand efficiently. Because price caps impede price rationing, I assume efficient non-price rationing. This rationing rule yields the canonical inverse residual demand function. Given an inverse residual demand function, the price cap applies: A firm gets paid the minimum of its inverse residual demand and its price cap. Intuitively, a price cap makes the inverse residual demand function flat at the price cap.

I find that with asymmetric price caps, the firms produce unequal quantities. Even an arbitrarily small asymmetry in the price caps may change the allocation of production discontinuously. Moreover, there is a trade-off between total quantity and production efficiency when only one price cap binds. Changing the binding price cap to increase the total quantity makes the production more unequal—less efficient. Thus, increasing the traded quantity is not equivalent to increasing the welfare anymore.

The good news is that asymmetric price caps that increase the welfare (compared to no price caps) generally exist: In the Cournot-Nash equilibrium without price caps, the firms' quantities are identical, so production is efficient. When one firm gets a price cap marginally below the Cournot-Nash equilibrium price, the total quantity increases, but, as the firms have the same marginal cost, the marginal distortion of production has no effect.

1. See the German law, Postgesetz (PostG) § 19 "Genehmigungsbedürftige Entgelte."

2. See Subsection 4.8.

3. See, for example, California. Executive Order N-44-20 (Executive Department, State of California, 2020) completes the Penal Code Section 396. Because an emergency has been declared, sellers of essential goods were forbidden to increase their prices by more than 10%, unless they were only passing on additional costs. Sellers that only started selling the goods after the emergency had been declared, were not allowed to charge prices exceeding their cost by more than 50%.

4. See Nakamura and Steinsson (2011) for anecdotal evidence of firms' committing themselves to not increase their prices.

These results are driven by two assumptions. *Efficient rationing* means that the allocation maximizes the consumer surplus, even if the price does not clear the market. Intuitively, there might be an unregulated resale market that allocates the goods to those with the highest willingness to pay. This assumption pins down the firms' inverse residual demand functions.

*Strategic substitutability* is a standard regularity assumption. It means that, in the absence of price caps, if one firm increases its quantity, it is optimal for the other firm to decrease its quantity. A log-concave inverse demand function is an assumption on the model primitives that guarantees strategic substitutability.

Strategic substitutability provides the basis for the trade-off between quantity and production efficiency that changing the price cap entails: Let a binding price cap be set to relieve a firm of its effect on its price, such that it is no longer optimal for the firm to withhold production and it produces more than without the price cap. If the firm's binding price cap is changed to incentivize it to expand its quantity and the other firm's price cap does not bind, the other firm's production is crowded out because of strategic substitutability. Because the firm with the binding price cap produces a larger quantity, this means that expensive production crowds out cheap production.

In a similar setting with symmetric price caps, there is a continuum of pure-strategy Nash equilibria (Okumura, 2017). In each of these equilibria, the total quantity in the market is the same, only the split between the duopolists differs. In particular, the continuum contains a symmetric equilibrium with efficient production.

My contribution to Okumura (2017) is showing that small asymmetries in the price caps can have big effects. The continuum of equilibria exists only because the price caps are symmetric: In each equilibrium of the continuum, both firms' price caps just bind. Therefore, both firms are at a discontinuous drop of their marginal profit: For any higher quantity, the price cap does not bind anymore and the firm depresses its price when increasing its quantity, so increasing the quantity causes a loss for all inframarginal units. For any lower quantity, the firm does not affect its price because the price cap strictly binds. The appearance of the inframarginal loss makes the marginal profit drop. As both firms are at the drop in their marginal profit, there is some leeway in the firms' optimality conditions, so a continuum of quantities satisfies them.

With the slightest asymmetry, however, the continuum of equilibria collapses to a single equilibrium. The reason is that asymmetric price caps cannot just bind at the same total quantity. Thus, the firms are not both at the drops in their marginal profit. The firm whose price cap is not just binding is not at the drop and has a (unique) optimal quantity; so there is no leeway, but a unique equilibrium. This equilibrium lies at one of the continuum's borders—so it has the most inefficient production.

The rest of the chapter is structured as follows. In Section 2, I explain the model and the price caps. In Section 3, I derive the unique pure-strategy Nash equilibria and their welfare implications. Subsection 3.5 deals with the special case of linear demand and quadratic cost, in which observable data might help to evaluate the price caps. In Section 4, I discuss extensions, generalizations, and applications. Extensions are the consumer surplus as an alternative objective (Subsection 4.1), the special case of constant marginal cost (Subsection 4.2), and the relationship with the sequential Stackelberg competition, which offers the novel interpretation that the Stackelberg leader commits itself to a price cap instead of a quantity (Subsection 4.3). Generalizations are the case in which both firms have a price cap (Subsection 4.4), heterogeneous cost functions (Subsection 4.5), mixed-strategy Nash equilibria as another solution concept (Subsection 4.6), and proportional rationing as another rationing rule (Subsection 4.7). Applications are symmetric nominal price caps with vertically differentiated goods (Subsection 4.8), and research strands that might pick up the model (Subsection 4.9). In Section 5, I summarize and conclude. In the Appendix are the proofs (Appendix A) and the benchmark of price caps in a monopoly (Appendix B).

## 2 Model

There are two firms, 1 and 2. The firms engage in quantity competition: They simultaneously choose their quantities, non-negative real numbers, to maximize their profits. The quantity that firm 1 chooses is named $q_1$ and the quantity that firm 2 chooses is named $q_2$. Both firms produce the same good and have the same cost function $c(q_i)$. The cost function is two times continuously differentiable, and the marginal cost is weakly positive, $c'(\cdot) \geq 0$, and strictly increasing, $c''(\cdot) > 0$.

The market-clearing price for each total quantity is given by the inverse demand function $p(q)$. It is two times continuously differentiable and strictly decreasing wherever it is positive, $\forall q: \ p(q) > 0 \implies p'(q) < 0$. The market exists, that is, at least one firm wants to produce, $p(0) > c'(0)$, and the market is finite, $\exists x: \ p(x) < c'(x)$.

Another standard assumption is that the firms' quantities are strategic substitutes. An assumption on the inverse demand function that is sufficient for the quantities to be strategic substitutes is log-concavity (Amir, 1996). Furthermore, log-concavity is almost necessary for the existence of a pure-strategy Nash equilibrium in Cournot competition (Amir, 2005).[5] Therefore, I will refer to Assumption 1 directly as "strategic substitutability."

**Assumption 1** (Strategic substitutability)**.** The inverse demand function is strictly log-concave wherever it is positive,

$$\forall q \ : \ p(q) > 0 \implies \frac{\partial^2(\ln(p(q)))}{\partial q^2} < 0. \tag{1}$$

**Benchmark without Price Caps.** A firm's problem is to maximize its profit by choosing $q_i$, taking as given the $q_j$ that the other firm chooses,

$$\max_{q_i} \ \pi_i(q_i, q_j) = \max_{q_i} \ q_i \cdot p(q_i + q_j) - c(q_i). \tag{2}$$

A firm's marginal profit is the price it gets for the marginal unit less the inframarginal loss from depressing the price it gets for the inframarginal units and less the marginal cost,

$$\frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = p(q_i + q_j) + q_i \cdot p'(q_i + q_j) - c'(q_i). \tag{3}$$

5. "Almost" because the set of quantities for which the inverse demand function has to be log-concave can be restricted. Also, if the marginal cost is strictly increasing, this adds some wiggle room for the inverse demand function.

The strategic substitutability assumption implies that the firms' profit functions are strictly quasi-concave,

$$\forall q_j \;:\; \frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = 0 \implies \frac{\partial^2 \pi_i(q_i, q_j)}{\partial q_i^2} < 0. \tag{4}$$

Thus, the best response is unique.

As mentioned above, strategic substitutability also implies that each firm's best response function is decreasing in the quantity of the other firm as long as the best response is positive, so

$$\forall q_j \;:\; \frac{\partial \pi_i(q_i, q_j)}{\partial q_i} = 0 \implies \frac{\partial^2 \pi_i(q_i, q_j)}{\partial q_i \, \partial q_j} < 0. \tag{5}$$

Together with the other standard assumptions, strategic substitutability implies the existence and uniqueness of the pure-strategy Cournot-Nash equilibrium.[6]

The unique pure-strategy Cournot-Nash equilibrium is symmetric. Both firms choose $q^C$ such that the first-order condition $p(2q^C) + q^C \cdot p'(2q^C) - c'(q^C) \stackrel{!}{=} 0$ is satisfied. In the following, I will refer to this equilibrium as the *Cournot-Nash equilibrium*, to the equilibrium quantity $q^C$ as the *Cournot-Nash quantity*, and to the equilibrium price $p^C \equiv p(2q^C)$ as the *Cournot-Nash price*.

**Price Caps.** The novel element that I introduce to the Cournot setting are asymmetric price caps. In the main part, only firm 1 has a price cap, $\bar{p}$. The case in which firm 2 also has a price cap is deferred to Subsection 4.4. The difference that firm 1's price cap makes, can be split into how the inverse residual demand functions (the price that a firm expects when producing a certain quantity while taking the other firm's quantity as given) are determined and, given its inverse residual demand function, at which price firm 1 can sell its good.

Before considering the price cap's effect, it is helpful to reconsider the standard Cournot case to better understand the meaning and derivation of the inverse residual demand function without price caps. When firm 1 makes its quantity choice, it expects its inverse residual demand to depend on its own choice and on the choice of its opponent, $p_1(q_1, q_2)$. Moreover, it rightly expects that $p_1(q_1, q_2) \equiv p(q_1 + q_2)$. The reason for this equivalence is that price rationing is efficient: After the firms have chosen their quantities, as the price is free to adjust, the price that clears the market will realize. Firm 1 anticipates this when choosing its quantity. Thus, firm 1's price when producing its first marginal unit is $p(q_2)$. Analogously, the price when producing $q_1$ units is $p(q_1 + q_2)$. That is, firm 1's inverse residual demand curve is the inverse demand curve shifted to the left by $q_2$ units. Figure 1 illustrates this concept.

Given an inverse residual demand function, the only difference that a firm's price cap makes, is to flatten the inverse residual demand function at the price cap. Denote firm 1's inverse residual demand function if it had no price cap by $p_1(q_1, q_2)$. If firm 1 chooses a quantity that is so small such that $p_1(q_1, q_2)$ exceeds the price cap, firm 1 still receives only the price cap for its good. If firm 1 chooses a quantity that is so large such that $p_1(q_1, q_2)$ falls below the price cap, the price cap does not bind and

6. Existence is proven in Amir (1996). The uniqueness is a corollary of Proposition 1: The best response functions are continuous, and the slopes are strictly between -1 and 0. Thus, the best response functions can intersect at most once. Furthermore, the symmetry of the best response functions implies that the intersection is in positive quantities. There are no further equilibria involving corner solutions. When firm 1 plays a quantity of 0, then firm 2 plays the monopoly quantity, but because the monopoly price is larger than the marginal cost of the first unit, firm 1's best response to the monopoly quantity is positive. Figure 12 is a sketch of the best response functions in the benchmark (among other things).

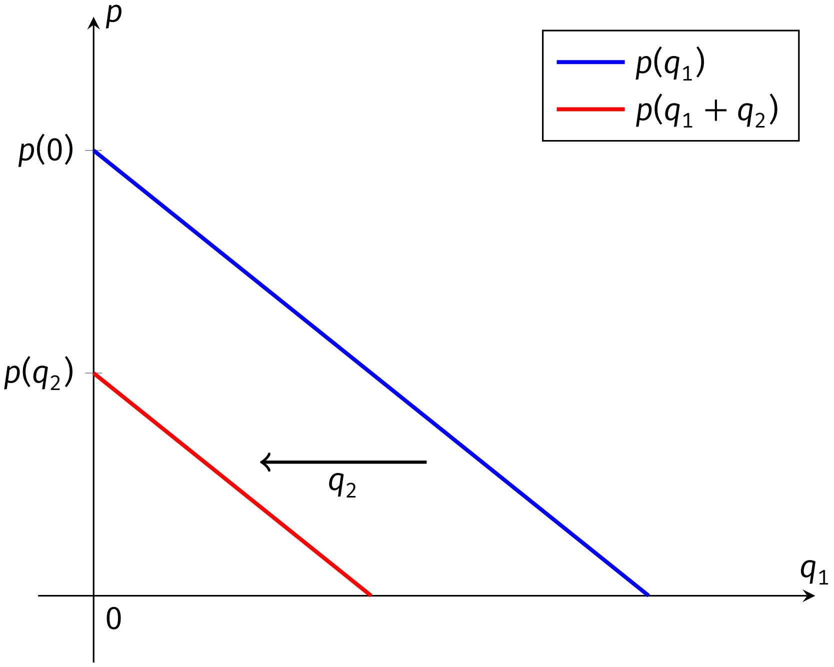


**Figure 1.** Firm 1's inverse residual demand curve is the inverse demand curve shifted to the left by $q_2$ units.

has no effect. Thus, the inverse residual demand function is $p_1(\bar{p}, q_1, q_2) = \min\{\bar{p},\ p_1(q_1, q_2)\}$. Figure 2 illustrates this concept.

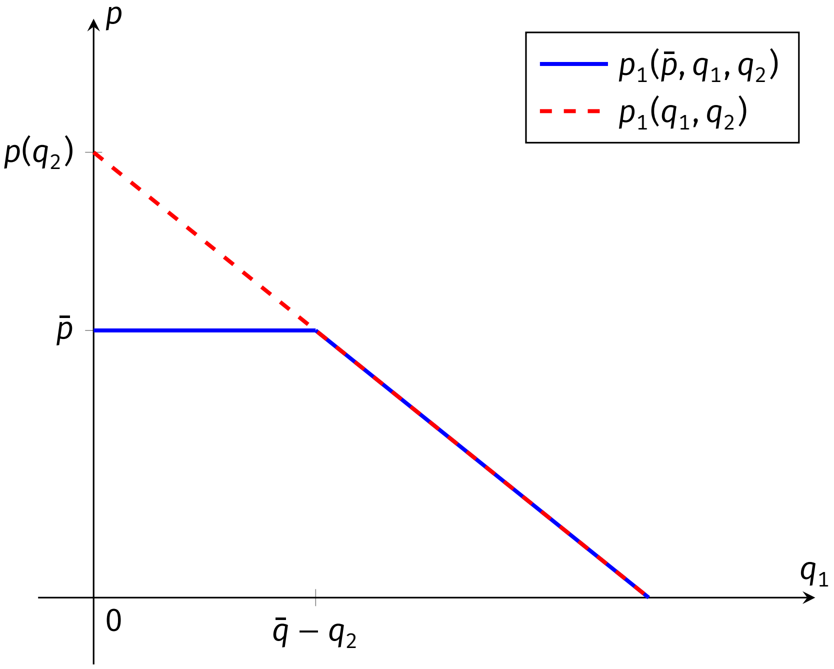


**Figure 2.** Given an inverse residual demand curve without a price cap, the price cap flattens it at the top. If the inverse demand would exceed the price cap, the firm still only gets paid the price cap.

To close the model, it only remains to determine how the asymmetric price cap affects the original inverse residual demand functions. For firm 1, there is no difference to the standard Cournot case. Because firm 2 has no price cap, its quantity is rationed efficiently as in the standard Cournot case. To simplify notation, I will denote the total quantity that makes firm 1's price cap just bind by $\bar{q} \equiv p^{-1}(\bar{p})$.

Thus,

$$p_1(\bar{p}, q_1, q_2) = \begin{cases} \bar{p} & \text{if } q_1 < \bar{q} - q_2 \\ p(q_1 + q_2) & \text{if } q_1 \geq \bar{q} - q_2. \end{cases} \tag{6}$$

The inverse residual demand function of firm 2 is not immediate. As firm 1 has a price cap, its quantity will be sold below the market-clearing price if firm 2 chooses a low enough quantity. Thus, it depends on a (non-price) rationing rule, which parts of the demand function get satisfied and which remain for firm 2.[7]

In the literature, there are two different rationing rules, the first of which is efficient rationing. I assume this rationing rule in the main part. It says that the quantities always get rationed to maximize the consumer surplus. Thus, the inverse residual demand is the left-shifted inverse demand; the same as in the standard Cournot case in Figure 1. Intuitively, when microfounding the demand function as a continuum of consumers with different valuations, this means that the consumers get served in the decreasing order of their valuations; for example because there is an unregulated secondary market. Expressed in terms of firm 2's problem, efficient rationing is summarized in Assumption 2.

**Assumption 2** (Efficient rationing)**.** Firm 2's inverse residual demand function is $p(q_1 + q_2)$.

The other usual rationing rule in the literature is proportional rationing (see Subsection 4.7).

## 3 Equilibria and Welfare Analysis

In this section, I solve for the unique pure-strategy Nash equilibrium for each price cap of firm 1. See Subsection 4.6 for a condition under which no mixed-strategy Nash equilibria exists. For readability, I will refer to the pure-strategy Nash equilibrium simply as "equilibrium."

I restrict the range of permissible price caps from below (exclusively) by the marginal cost of the first unit and from above (inclusively) by the Cournot-Nash price, $\bar{p} \in (c'(0), p^C]$. A lower price cap would exclude firm 1 from the market, and a higher price cap would have no effect.

### 3.1 The Firms' Maximization Problems

Due to efficient rationing, firm 2's profit maximization problem is the same as in the benchmark. Whenever firm 2's best response is positive, it is indirectly defined by the solution to its first-order condition,

$$BR_2(q_1) = q_2^*(q_1): \quad p(q_1 + q_2^*(q_1)) + q_2^*(q_1) \cdot p'(q_1 + q_2^*(q_1)) - c'(q_2^*(q_1)) = 0. \tag{7}$$

7. When the price caps are symmetric (as in Okumura, 2017), the inverse residual demand functions are independent of the rationing rule. To see this, consider the interpretation of the demand function as mapping prices into measures of consumers whose willingness to pay exceeds the price. As both firms have the same price cap, even consumers with a larger willingness to pay never pay more than the symmetric price cap. Thus, the inverse demand function could be replaced with the same function, but flattened at the price cap. When the price cap binds, only consumers whose willingness to pay equals the price cap are served, so it does not matter who exactly gets served; the firms' inverse residual demand functions remain the same.

To improve the readability, I suppress the (irrelevant) corner solution $q_2 = 0$, which is only optimal if $p(q_1) - c'(0) \leq 0$, that is, if firm 1 supplies enough to serve the whole market at the marginal cost of the first unit or less.

Firm 1 maximizes its profit using the residual demand function from above,

$$\max_{q_1} \pi_1(q_1, q_2) = \max_{q_1} \begin{cases} q_1 \cdot \bar{p} - c(q_1) & \text{if } q_1 < \bar{q} - q_2 \\ q_1 \cdot p(q_1 + q_2) - c(q_1) & \text{if } q_1 \geq \bar{q} - q_2. \end{cases} \tag{8}$$

The marginal profit is, expressed as the right-derivative at the drop[8] at $q_1 = \bar{q} - q_2$,

$$\frac{\partial_+ \pi_1(q_1, q_2)}{\partial q_1} = \begin{cases} \bar{p} - c'(q_1) & \text{if } q_1 < \bar{q} - q_2 \\ p(q_1 + q_2) + q_1 \cdot p'(q_1 + q_2) - c'(q_1) & \text{if } q_1 \geq \bar{q} - q_2. \end{cases} \tag{9}$$

The composition of the marginal profit is illustrated in Figure 3. Firm 1's marginal profit consists of two parts. For $q_1 < \bar{q} - q_2$, the price cap strictly binds, so firm 1 is a price-taker. Because of the price cap, the marginal revenue is constantly $\bar{p}$ and the marginal profit is strictly decreasing as the marginal cost is strictly increasing. The root of this part is $(c')^{-1}(\bar{p})$. For $q_1 \geq \bar{q} - q_2$, the price cap just binds or does not bind, so firm 1's marginal profit is that of a standard Cournot duopolist. The own effect on the price—the inframarginal loss—from increasing the quantity is $q_1 \cdot p'(\bar{q})$. It is the inframarginal loss that makes imperfectly competitive firms withhold quantity (and eliminating the inframarginal loss is the reason why price caps work). The root of this part is $BR_2(q_2)$. As the price and the marginal cost are continuous at $q_1 = \bar{q} - q_2$, firm 1's marginal profit drops by the inframarginal loss, $q_1 \cdot p'(\bar{q})$.[9]

Firm 1's profit function is strictly quasi-concave in $q_1$: The profit function is continuous and the marginal profit functions of both the price-taker and the standard Cournot duopolist intersect zero exactly once and from above. The price-taker's because it is strictly decreasing and the standard Cournot duopolist's because it is the benchmark. Because firm 1's marginal profit drops downwards at $q_1 + q_2 = \bar{q}$, the combined marginal profit does not contain both intersections with zero but exactly one (possibly at the drop).

As firm 1's profit function is strictly quasi-concave in $q_1$, the intersection of the marginal profit and zero determines firm 1's best response if it is positive. The corner solution is optimal in the same case as for firm 2 and is, as well, ignored. The interior best responses of firm 1 depend both on the opponent's choice and the price cap, and they can be grouped in three cases, which also occur when applying a price cap to a monopoly (see Appendix B).

1) The intersection can be in the part where $q_1 > \bar{q} - q_2$. This case is illustrated in Figure 4. Even with the inframarginal loss after the price cap stops binding, the firm profits from expanding its quantity. As the price cap does not bind, it has no effect. Thus, the best response of a standard Cournot duopolist, $BR_2(q_2)$, is optimal if $BR_2(q_2) \geq \bar{q} - q_2$. In the monopoly, this case corresponds to high, ineffective price caps.

2) The intersection can be at the drop at $q_1 = \bar{q} - q_2$. This case is illustrated in Figure 5. As argued above, when the price cap stops binding, the inframarginal loss from depressing the price appears. In this

8. There is a strict drop for all price caps except for $\bar{p} = p^C$. In this case, there is only a kink. Nevertheless, I will refer to the "drop."

9. In fact, for all $q_1 > \bar{q} - q_2$, the marginal profit of the standard Cournot duopolist (the second part of the marginal profit function) is strictly less than the marginal profit of the price-taker (the first part of the marginal profit function): The price is lower, the inframarginal loss is negative, and the marginal cost is the same.

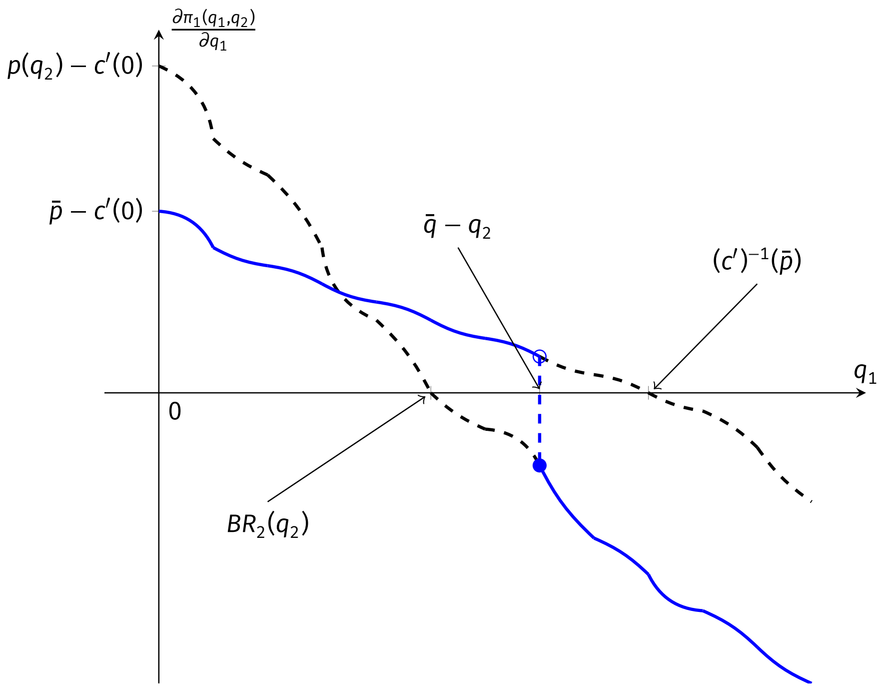


**Figure 3.** The marginal profit of the price-taker starts at $\bar{p} - c'(0)$ and its root is $(c')^{-1}(\bar{p})$. The marginal profit of the standard Cournot duopolist starts at $p(q_2) - c'(0)$ and its root is $BR_2(q_2)$. The marginal profit of firm 1 is in blue.

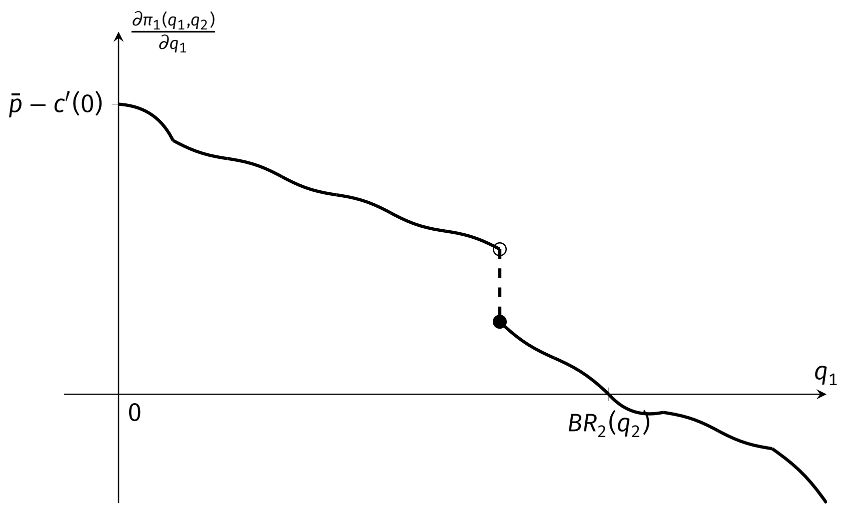


**Figure 4.** If the marginal profit is weakly positive after the drop, it is optimal for firm 1 to produce $BR_2(q_2)$.

case, the marginal profit from expanding its quantity is positive as long as the firm does not influence its price. But, the inframarginal loss is so large that the firm would lose from producing more when the price cap stops binding. Thus, $q_1 = \bar{q} - q_2$ is optimal whenever the marginal profit's drop starts in the weakly positive and ends in the weakly negative, that is, if $BR_2(q_2) \leq \bar{q} - q_2 \leq (c')^{-1}(\bar{p})$. In the monopoly, this case corresponds to intermediate price caps for which the inverse demand curve determines the quantity.

3) The intersection can be in the part where $q_1 < \bar{q} - q_2$. This case is illustrated in Figure 6. The price cap and the other firm's quantity are so low that firm 1's marginal cost reaches the price cap while it still binds: Firm 1 is a price-taker. Thus, the root of the marginal profit of the price-taker, $(c')^{-1}(\bar{p})$, is optimal whenever it is smaller than the quantity at which the price cap stops binding, that is, if

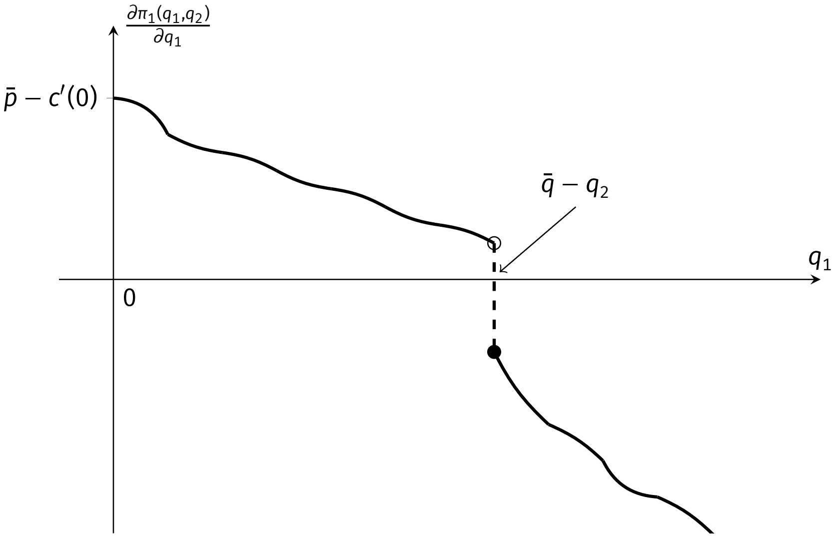


**Figure 5.** If the marginal profit is weakly positive above and weakly negative below the drop, it is optimal for firm 1 to produce $\bar{q} - q_2$.

$(c')^{-1}(\bar{p}) < \bar{q} - q_2$. In the monopoly, this case corresponds to low price caps for which the marginal cost curve determines the quantity.

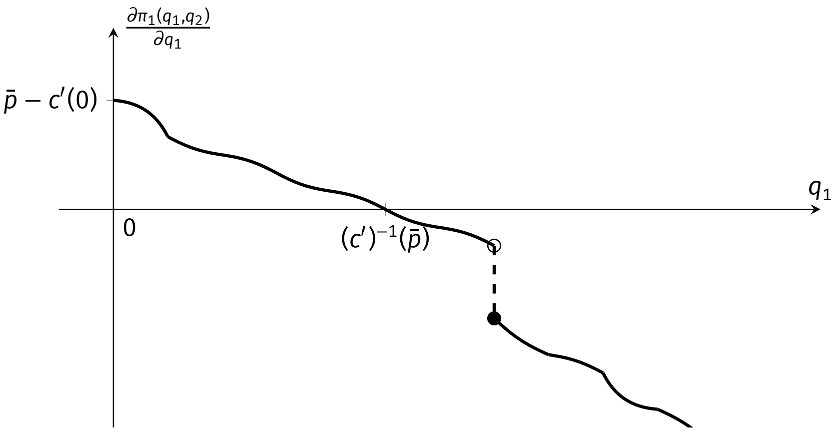


**Figure 6.** If the marginal profit is weakly negative above the drop, it is optimal for firm 1 to produce $(c')^{-1}(\bar{p})$.

These cases can be summarized in firm 1's best response function—ignoring corner solutions at 0—[10]

$$BR_1(q_2;\bar{p}) = \min\left\{\max\left\{BR_2(q_2),\ \bar{q} - q_2\right\},\ (c')^{-1}(\bar{p})\right\}. \tag{10}$$

You can see the best response function in Figure 7.

The intersections of the two firms' best response functions are the equilibria of the game. The properties of the equilibria depend on the part of firm 1's best response function in which the intersection is.

10. If you have kept track of the permutations, you might wonder whether this best response function yields the wrong value if $\bar{q} - q_2 < (c')^{-1}(\bar{p}) < BR_2(q_2)$ (it yields $(c')^{-1}(\bar{p})$, but $BR_2(q_2)$ maximizes the profit). The solution is that the above inequality cannot occur. Remember the fact that the standard Cournot duopolist part of the marginal profit is strictly smaller than the price-taker part for all $q_1 > \bar{q} - q_2$. Thus, if $\bar{q} - q_2 < BR_2(q_2)$, the root of the price-taker part has to be at an even higher quantity, so $\bar{q} - q_2 < BR_2(q_2) < (c')^{-1}(\bar{p})$. In this case, the best response function yields the right value: $BR_2(q_2)$.

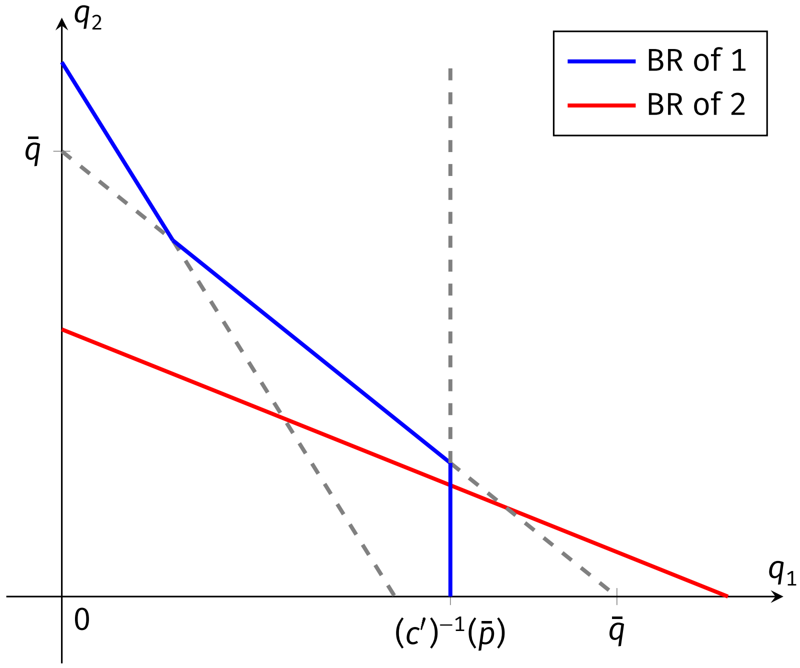


**Figure 7.** On the $x$-axis is the quantity of firm 1 and on the $y$-axis is the quantity of firm 2. The best response function of firm 2 (in red) is that of the standard Cournot duopolist. The best response function of firm 1 (in blue) consists of three parts. The left part is case 1), where the standard Cournot duopolist's best response is optimal. The middle part is case 2), where choosing the quantity at which the price cap just binds, $\bar{q} - q_2$, is optimal. The right part is case 3), where firm 1 is a price-taker and stops producing when the marginal cost reaches the price cap. For the comparative statics in the price cap, remember that the first part is independent of the price cap and that $\bar{q}$ and $(c')^{-1}(\bar{p})$ move in opposite directions. You can try out the comparative statics for a linear demand and quadratic cost in this Desmos Graphing Calculator graph: https://www.desmos.com/calculator/ritredhsbu (last accessed September 27, 2022).

The intersection cannot be in the interior of the first part of firm 1's best response function, $BR_2(q_2)$, because the total quantity would exceed $\bar{q}$. As Proposition 1 shows, then, at least one firm could profitably deviate. The idea is that both firms are standard Cournot duopolists if the total quantity exceeds $\bar{q}$. The slope of their best response functions is, then, strictly between $-1$ and 0 due to strategic substitutability. So if a firm produces more than in the Cournot-Nash equilibrium, its marginal profit gets negative unless the other firm reduces its quantity by even more; which means that the total quantity falls short of $\bar{q}$. This idea is illustrated in Figure 8.

**Proposition 1.** *If $q_1 + q_2 > \bar{q}$, then the marginal profit of at least one firm is strictly negative.*

*Proof.* The proof is in Appendix A, Subsection A.1 □

In the other two parts, there are equilibria. If the intersection is in the second part of $BR_1(q_2;\bar{p})$, then $q_1 + q_2 = \bar{q}$ and the price cap just binds in equilibrium. These are "clearing equilibria." If the intersection is in the third part of $BR_1(q_2;\bar{p})$, then $q_1 + q_2 < \bar{q}$ and the price cap strictly binds in equilibrium. These are "rationing equilibria." In the following, I explain the names and show that if the price cap is above a cutoff, $\kappa$, the unique equilibrium is a clearing equilibrium and that if the price cap is below the cutoff $\kappa$, the unique equilibrium is a rationing equilibrium.

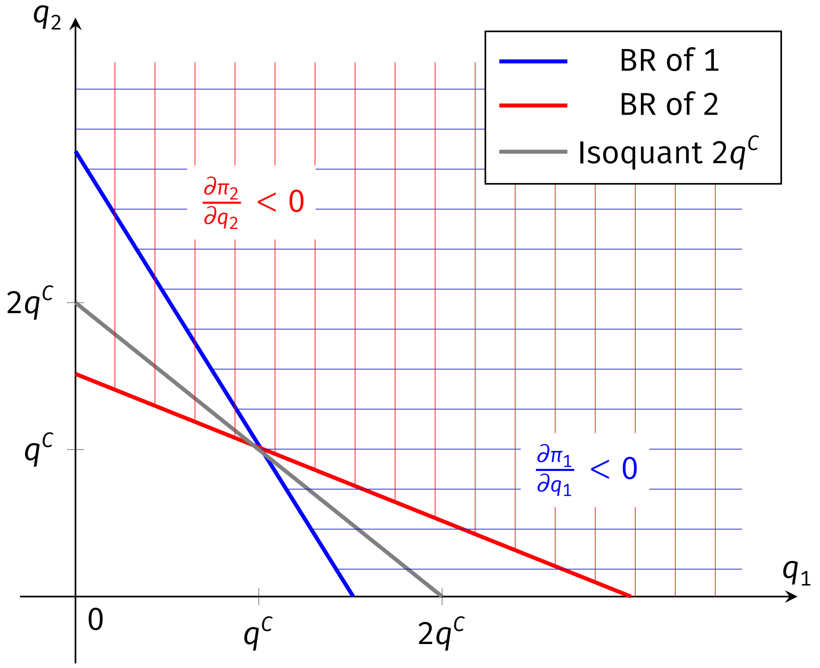


**Figure 8.** On the $x$-axis is the quantity of firm 1 and on the $y$-axis is the quantity of firm 2. The gray isoquant shows the Cournot-Nash total quantity and has a slope of -1. The best response function of firm 2 (red) is everywhere flatter, and the best response function of firm 1 (blue) is everywhere steeper (because the axes are inverted). Above the best response function, firm 2's marginal profit is negative and to the right of the best response function, firm 1's marginal profit is negative because the profit functions are strictly quasi-concave.

## 3.2 Clearing Equilibria

For each price cap, there is a unique candidate for a clearing equilibrium. The reason is that the equilibrium condition $q_1 + q_2 = \bar{q}$ leaves only one split of $\bar{q}$ into $q_1^*(\bar{q})$ and $q_2^*(\bar{q})$ such that the first-order condition of firm 2 is satisfied. This candidate is an equilibrium if its suggested quantity is optimal for firm 1; that is, if the marginal profit is weakly positive above and weakly negative below the drop (as in Figure 5). Theorem 1 proves that this candidate is the unique equilibrium if the price cap is above the cutoff $\kappa$.

**Theorem 1 (Clearing Equilibria).** *The quantities*

$$q_1^*(\bar{q}) \;=\; \bar{q} - q_2^*(\bar{q}) \quad \text{and} \quad q_2^*(\bar{q}) \;:\; p(\bar{q}) + q_2^*(\bar{q}) \cdot p'(\bar{q}) - c'(q_2^*(\bar{q})) \;=\; 0 \tag{11}$$

*have the properties that*

*(i)* $q_1^*(\bar{q}) \geq q^C \geq q_2^*(\bar{q})$, *with strict inequalities for* $\bar{q} > 2q^C$.

*(ii)* $q_1^*(\bar{q})$ *is strictly increasing in* $\bar{q}$ *and* $q_2^*(\bar{q})$ *is strictly decreasing in* $\bar{q}$.

*There is a cutoff* $\kappa \in (c'(0), p^C)$ *that is indirectly defined by*

$$\kappa \;:\; \kappa - c'\left(q_1^*(p^{-1}(\kappa))\right) \;=\; 0. \tag{12}$$

*It has the properties that*

*(iii)* $q_1^*(\bar{q})$ *and* $q_2^*(\bar{q})$ *are the unique equilibrium for all* $\bar{p} \in [\kappa, p^C]$.

*(iv)* $q_1^*(\bar{q})$ *and* $q_2^*(\bar{q})$ *are no equilibrium for all* $\bar{p} \in (c'(0), \kappa)$.

*Proof.* The proof is in Appendix A, Subsection A.2. □

I call these equilibria "clearing" because there is no excess demand, as the price cap just binds; both firms receive the same price. Theorem 1 contains the message of this chapter: The asymmetric price cap makes the production asymmetric and, thus, inefficient, and the more so the more it increases the total quantity.

To analyze the impact of the inefficient production on the usefulness of asymmetric price caps, consider the (utilitarian) *welfare*. It is the sum of the consumer surplus and the firms' profits. Formally, the welfare is the area between the inverse demand curve and the marginal cost curves,

$$W(\bar{q}) \;=\; \int_0^{\bar{q}} p(x)\ dx - c(q_1^*(\bar{q})) - c(q_2^*(\bar{q})). \tag{13}$$

The derivative of the welfare with respect to the price cap shows the trade-off between the total quantity and the production efficiency. When the price cap is decreased to increase the total quantity, firm 1 increases its quantity, which crowds out some of firm 2's quantity. Thus, increasing the total quantity affects the welfare in two ways: The net increase in the total quantity is the (weakly) positive quantity effect, and the crowding out of firm 2's quantity is the (weakly) negative production efficiency effect. Formally, the derivative is[11]

$$\frac{\partial W(\bar{q})}{\partial \bar{q}} \;=\; \underbrace{\bar{p} - c'(q_1^*(\bar{q}))}_{\text{quantity}} \underbrace{-\left(-\frac{\partial q_2^*(\bar{q})}{\partial \bar{q}}\right) \cdot \big(c'(q_1^*(\bar{q})) - c'(q_2^*(\bar{q}))\big)}_{\text{production efficiency}}, \tag{14}$$

where I adjusted the minuses to make all brackets weakly positive and used that $q_1^*(\bar{q}) = \bar{q} - q_2^*(\bar{q})$.

The quantity effect is (weakly) positive because firm 1's increasing the total quantity increases the welfare: The marginal contribution to the welfare is $\bar{p}$ and the social marginal cost is that firm 1 incurs additional costs of $c'(q_1^*(\bar{q}))$. If the price cap is strictly above the cutoff, the difference is strictly positive.

The production efficiency effect is (weakly) negative because firm 1 produces weakly more and has, thus, a weakly larger marginal cost. Therefore, crowding out cheaper quantity from firm 2 is socially wasteful. The production efficiency effect is the product of two terms: By how much firm 2 reduces its quantity—the stronger firm 2 reacts to a change in the price cap, the more firm 1 has to compensate—and how unequal the marginal cost is already—the larger the difference is, the more socially costly is the compensation.

The lower the price cap is, the more tends the social loss from a more inefficient production to outweigh the social gain from more production. When the price cap is lower, the positive marginal effect of a larger total quantity gets smaller as the social benefit and the social cost of a larger quantity converge. The negative marginal effect from a less efficient production tends to get larger, as the marginal cost are more unequal. This tendency might be locally overturned because the other factor—how much firm 2 reacts to a marginal change in the price cap—might be smaller for some price caps.[12] Thus, the

11. I take the derivative with respect to the total equilibrium quantity $\bar{q}$ to improve readability. To get the marginal effect in terms of a one unit decrease of the price cap, multiply with the negative term $\frac{\partial \bar{q}}{\partial \bar{p}}$.

12. With linear demand and quadratic cost, firm 2's reaction is constant in the price cap. Then, the negative effect from a more inefficient production is monotone in the price cap (see Subsection 3.5).

welfare effect of a marginal change in the price cap is generally ambiguous—except for the extreme price caps.

When the price cap equals the Cournot-Nash price, the production efficiency effect vanishes because both firms produce the same quantity and have the same marginal cost. Thus, shifting production marginally from firm 2 to firm 1 does not change the social cost. Evaluating the derivative at the Cournot-Nash price yields

$$\left.\frac{\partial_+ W(\bar{q})}{\partial \bar{q}}\right|_{\bar{q}=2q^C} = p^C - c'(q^C) > 0. \tag{15}$$

The inequality follows from the Cournot-Nash equilibrium condition. This result means that introducing asymmetric price caps just below the Cournot-Nash price always increases the welfare.

When the price cap equals the cutoff between clearing and rationing equilibria, $\kappa$, the quantity effect vanishes: At the cutoff, the marginal cost of firm 1 equals the social marginal benefit. Thus, only the negative production efficiency effect remains,

$$\left.\frac{\partial_- W(\bar{q})}{\partial \bar{q}}\right|_{\bar{q}=p^{-1}(\kappa)} = -\left(-\frac{\partial_- q_2^*(p^{-1}(\kappa))}{\partial \bar{q}}\right) \cdot \left(c'(q_1^*(p^{-1}(\kappa))) - c'(q_2^*(p^{-1}(\kappa)))\right) < 0. \tag{16}$$

This result means that it is not innocuous that the price cap distorts the production: At the cutoff, a marginally higher price cap would increase the welfare although the total quantity would decrease. So, the regulator should not rely on the total quantity alone to evaluate asymmetric price caps.

### 3.3 The Cutoff

Theorem 1 has only shown that the cutoff, $\kappa$, lies in the range of permissible price caps, $(c'(0), p^C]$. In this subsection, I will explore the nature of the cutoff and present bounds.

The reason for the existence of the cutoff is the monotonicity of the comparative statics: The lower the price cap is, the higher is the clearing equilibrium candidate quantity of firm 1, which in turn means that the marginal profit above the drop is lower. As long as the price cap is strictly above the cutoff, the marginal profit above the drop is positive. When the price cap is at the cutoff, $\kappa$, the marginal profit above the drop is exactly zero. For all lower price caps, firm 1's marginal profit above the drop is negative, so firm 1 wants to deviate to $(c')^{-1}(\bar{p})$ (see Figure 6).

As mentioned above, price regulation in a Cournot duopoly is closely related to price regulation in a monopoly (see Appendix B). The cutoff, $\kappa$, is analogous to the perfectly competitive price in the monopoly: As long as the price cap is above the cutoff, the quantity of firm 1 is determined by the intersection of the marginal revenue and the inverse residual demand curve. When the price cap is below the cutoff, its quantity is determined by the intersection of the marginal revenue and the marginal cost curve.

Because there is another firm in the market, the cutoff, $\kappa$, lies strictly above the perfectly competitive price as Proposition 2 shows.[13] There cannot be a clearing equilibrium with the perfectly competitive price as the price cap. The reason is that both firms would have to supply half of the perfectly competitive

13. In a duopoly with symmetric cost, the perfectly competitive price is given by the intersection of the inverse demand curve and the social marginal cost curve. The social marginal cost is each firm's marginal cost when splitting the total quantity equally—when producing efficiently.

quantity, but firm 2 profits from deviating to a lower quantity. This result means that it is impossible to achieve full efficiency with asymmetric price caps.

Concerns that the cutoff could be (arbitrarily) close to the Cournot-Nash equilibrium and that clearing equilibria could be, thus, not particularly interesting, are unnecessary. Proposition 2 also shows that the cutoff lies strictly below the Stackelberg equilibrium price, if the Stackelberg leader's profit function is strictly quasi-concave.[14]

**Proposition 2.** *Define the competitive price, $p^W$, as the price at which the inverse demand curve, $p(q)$, and the social marginal cost curve, $c'(\frac{q}{2})$, intersect.*

*Define $p^S$ as the unique Stackelberg equilibrium price if the Stackelberg leader's profit function is strictly quasi-concave.*

*It is true that $p^W < \kappa < p^S$.*

*Proof.* The proof is in Appendix A, Subsection A.3. □

### 3.4 Rationing Equilibria

When the price cap is below the cutoff, $\kappa$, the total quantity has to fall short of $\bar{q}$ in all equilibria: Theorem 1 shows that the unique candidate for an equilibrium with total quantity $\bar{q}$ is no equilibrium and Proposition 1 shows that no equilibrium has a total quantity exceeding $\bar{q}$.

Because the total quantity is below $\bar{q}$ in equilibrium, firm 1's price cap binds strictly, so it acts as a price-taker: Firm 1's equilibrium strategy is to produce until the marginal cost equals the price cap. Firm 2's equilibrium strategy is the solution to the monopolist's problem in the market for residual demand. Theorem 2 formally summarizes the rationing equilibrium strategies. Because $\bar{q}$ has no particular meaning in the rationing equilibria and to distinguish them from the clearing equilibria, I denote the equilibrium strategies as functions of $\bar{p}$.

**Theorem 2 (Rationing Equilibria).** *If $\bar{p} \in (c'(0), \kappa)$, the only pure-strategy Nash equilibrium is*

$$q_1^*(\bar{p}) \;=\; (c')^{-1}(\bar{p}) \quad \textit{and} \tag{17}$$

$$q_2^*(\bar{p}) \;:\; p\Big(q_1^*(\bar{p}) + q_2^*(\bar{p})\Big) + q_2^*(\bar{p}) \cdot p'\Big(q_1^*(\bar{p}) + q_2^*(\bar{p})\Big) - c'(q_2^*(\bar{p})) \overset{!}{=} 0. \tag{18}$$

*In the limit of $\bar{p} \to \kappa$, the equilibrium converges to the clearing equilibrium presented in Theorem 1.*

*$q_1^*(\bar{p})$ is strictly increasing in $\bar{p}$ and $q_2^*(\bar{p})$ is strictly decreasing in $\bar{p}$. The total quantity $q_1^*(\bar{p}) + q_2^*(\bar{p})$ is strictly increasing in $\bar{p}$.*

*Proof.* The proof is in Appendix A, Subsection A.4. □

In the rationing equilibria, there is rationing in the equilibrium. Firm 1 sells its quantity at the price cap, and firm 2 does not want to serve all the excess demand at that price. Thus, firm 2 sells its quantity at a higher price that clears the market for the residual demand.

14. The qualification assures that the Stackelberg equilibrium price is unique and determined by the Stackelberg leader's first-order condition, which is used to prove the inequality. A sufficient condition for strict quasi-concavity is, for example, a weakly concave inverse demand function. For a short definition of the sequential Stackelberg quantity competition model, see Subsection 4.3.

The comparative statics are reversed compared to the clearing equilibria. When the price cap decreases, firm 1, being a price-taker, produces a smaller quantity in equilibrium. Firm 2 produces a larger quantity as its market gets larger because of strategic substitutability. The total quantity, however, decreases.

Table 1 summarizes the equilibria and their comparative statics that go in opposite directions.

**Table 1.** This table summarizes the equilibrium quantities and the respective comparative statics in the two different types of equilibria.

| | Price caps | Firm 1 | Firm 2 | Total quantity |
|---|---|---|---|---|
| Clearing equilibria (Theorem 1) | $p^C > \bar{p} \geq \kappa$ | $q_1^* = \bar{q} - q_2^*$ <br> $\frac{\partial q_1^*}{\partial \bar{p}} < 0$ | $q_2^*$ solves FOC <br> $\frac{\partial q_2^*}{\partial \bar{p}} > 0$ | $\frac{\partial (q_1^*+q_2^*)}{\partial \bar{p}} < 0$ |
| Rationing equilibria (Theorem 2) | $\kappa > \bar{p} > c'(0)$ | $q_1^* = (c')^{-1}(\bar{p})$ <br> $\frac{\partial q_1^*}{\partial \bar{p}} > 0$ | $q_2^*$ solves FOC <br> $\frac{\partial q_2^*}{\partial \bar{p}} < 0$ | $\frac{\partial (q_1^*+q_2^*)}{\partial \bar{p}} > 0$ |

The reversal of the comparative statics is no coincidence: Clearing and rationing equilibria are symmetric to each other around the cutoff. Whenever the total quantity is the same in a clearing and in a rationing equilibrium, both firms' quantities are identical. The reason is that firm 2's best response is unique for each total quantity. Furthermore, the total quantity is monotone and continuous in the price cap both within the clearing and the rationing equilibria. Lemma 1 proves the symmetry.

**Lemma 1.** *Define $\bar{p}_B$ as the price cap for which the total quantity in the rationing equilibrium is equal to the Cournot-Nash quantity, $\bar{p}_B : q_1^*(\bar{p}_B) + q_2^*(\bar{p}_B) = 2q^C$. It holds that $c'(0) < \bar{p}_B < \kappa$.*

*There is a monotone bijection between the clearing equilibria and the rationing equilibria. For each price cap $\bar{p}_c \in (\kappa, p^C]$, there is exactly one price cap $\bar{p}_r \in [\bar{p}_B, \kappa)$ such that the equilibrium quantities of the firms are the same:*

$$q_1^*(\bar{q}_c) = q_1^*(\bar{p}_r) \qquad \text{and} \qquad q_2^*(\bar{q}_c) = q_2^*(\bar{p}_r). \tag{19}$$

*Proof.* The proof is in Appendix A, Subsection A.5. □

Because equal equilibrium quantities imply equal welfare, the symmetry result extends to the welfare: The welfare effects of a decreasing price cap are reversed for the rationing equilibria. As long as the total quantity exceeds the Cournot-Nash quantity, a lower price cap reduces the total quantity but makes the production more efficient.

If the price cap is below $\bar{p}_B$, there is no trade-off between total quantity and production efficiency anymore. A lower price cap means that the total quantity decreases and that the production gets less efficient because firm 2 produces already initially more than firm 1 and its quantity increases further. When the price cap approaches $c'(0)$, the welfare goes to the welfare in a monopoly. Anyway, the welfare with any asymmetric price cap below $\bar{p}_B$ is lower than the welfare in the Cournot-Nash equilibrium without a price cap.

Due to the symmetry, the welfare takes a local minimum at the cutoff. As a decreasing price cap decreases the welfare in the clearing equilibria just above the cutoff, a decreasing price cap increases the welfare in the rationing equilibria just below the cutoff.

Even if the regulator that sets the price cap knows nothing about the functional forms, she might be able to observe whether a price cap is at the cutoff: The market is at the brink of segmenting into a regulated low-price part and an unregulated high-price part. If the regulator observes this beginning segmentation, she should either decrease or increase the price cap.

### 3.5 Special Case of Linear Demand and Quadratic Cost

A special case often analyzed in the literature is a linear inverse demand, $p(q) = a - b \cdot q$, and quadratic cost, $c(q_i) = \frac{c}{2} \cdot q_i^2$. These functional forms eliminate many of the higher derivatives; so the best response functions are (piece-wise) linear. Below, there are plots of the equilibrium quantities (Figure 9) and of the consumer surplus and the welfare (Figure 10).

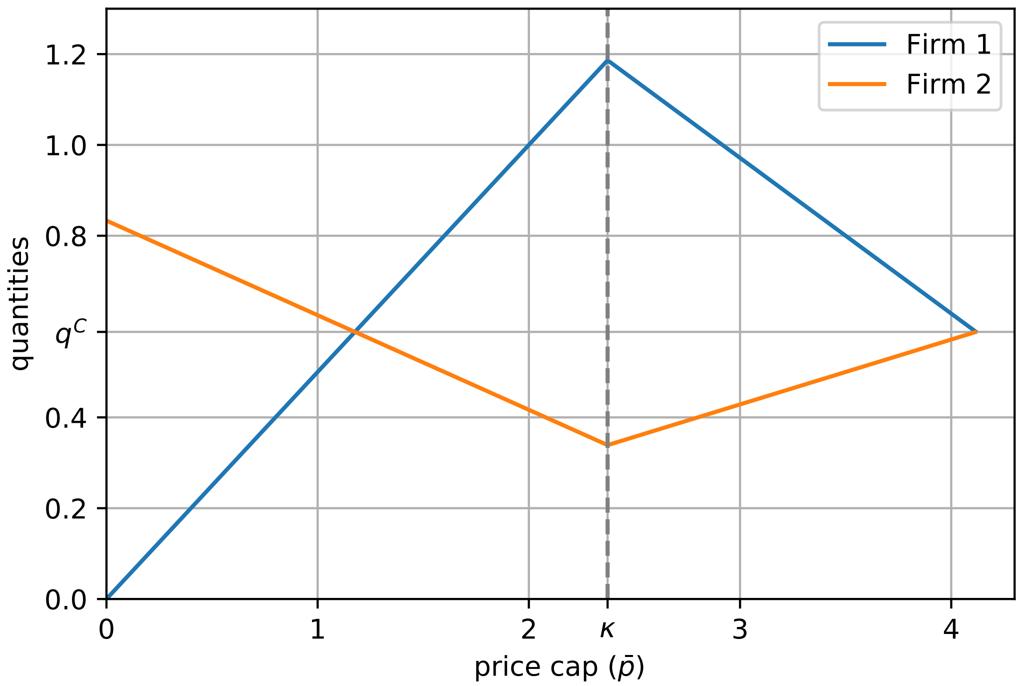


**Figure 9.** The equilibrium quantities of the firms when the cost functions are $c(q_i) = q_i^2$ and the inverse demand is $p(q) = 10 - 5 \cdot q$.

Figure 9, depicting the equilibrium quantities, shows that the comparative statics in clearing and in rationing equilibria are reversed. It also illustrates the symmetry result: Whenever firm 1's quantity is the same, so is firm 2's quantity.

Figure 10, depicting the consumer surplus and the welfare, shows some additional welfare results beyond the general case. The welfare is concave both within the clearing and in the rationing equilibria. The reason is that the functional forms make the production efficiency effect monotone in the price cap because the slope of firm 2's best response function is constant.

For all linear demand and quadratic cost functions, the welfare at the cutoff is larger than the welfare both in the Cournot-Nash equilibrium and in the unique Stackelberg equilibrium. This fact might help to evaluate the welfare effect of an asymmetric price cap. It implies that all clearing equilibria have

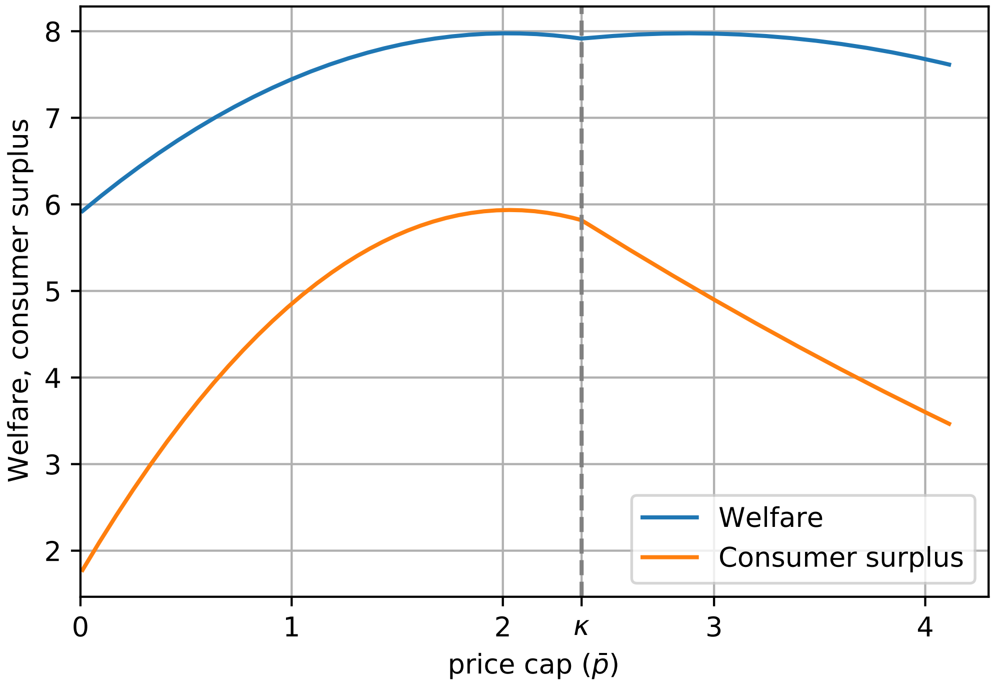


**Figure 10.** The welfare and consumer surplus in equilibrium when the cost functions are $c(q_i) = q_i^2$ and the inverse demand is $p(q) = 10 - 5 \cdot q$.

a higher welfare than the Cournot-Nash equilibrium. The symmetry then implies that the asymmetric price cap improves the welfare if and only if it increases the total quantity, which is the same as saying that it improves the welfare if and only if firm 1 produces a larger quantity than firm 2. Therefore, a regulator that chooses a price cap to maximize the total quantity, on the one hand, ends up in a local minimum of the welfare, but, on the other hand, still improves the welfare compared to the benchmark without a price cap.

## 4 Extensions, Generalizations, and Applications

In this section, I explore extensions and additional results, discuss generalizations and alternative assumptions, and present applications.

### 4.1 Consumer Surplus

This subsection deals with a different objective that the regulator might have: maximizing the *consumer surplus*. The consumer surplus—with efficient rationing—is defined as the area between the inverse demand curve and the prices up to the total quantity.

In clearing equilibria, the consumer surplus is

$$CS(\bar{q}) = \int_0^{\bar{q}} p(x) - \bar{p} \, dx. \tag{20}$$

It is unambiguously increasing in a decreasing price cap because the total quantity increases and the price decreases.

In contrast to the welfare, the consumer surplus is not symmetric between the rationing and the clearing equilibria. The reason is that while the firms' quantities are symmetric, the prices are not: The price of firm 1 is lower in the corresponding rationing equilibrium. Thus, compared to the consumer surplus in the corresponding clearing equilibrium, the consumers receive the price differential as an additional transfer from firm 1. Figure 11 illustrates the consumer surplus. The additional transfer is the rectangle $q_1^*(\bar{p}) \cdot (p(Q(\bar{p})) - \bar{p})$.

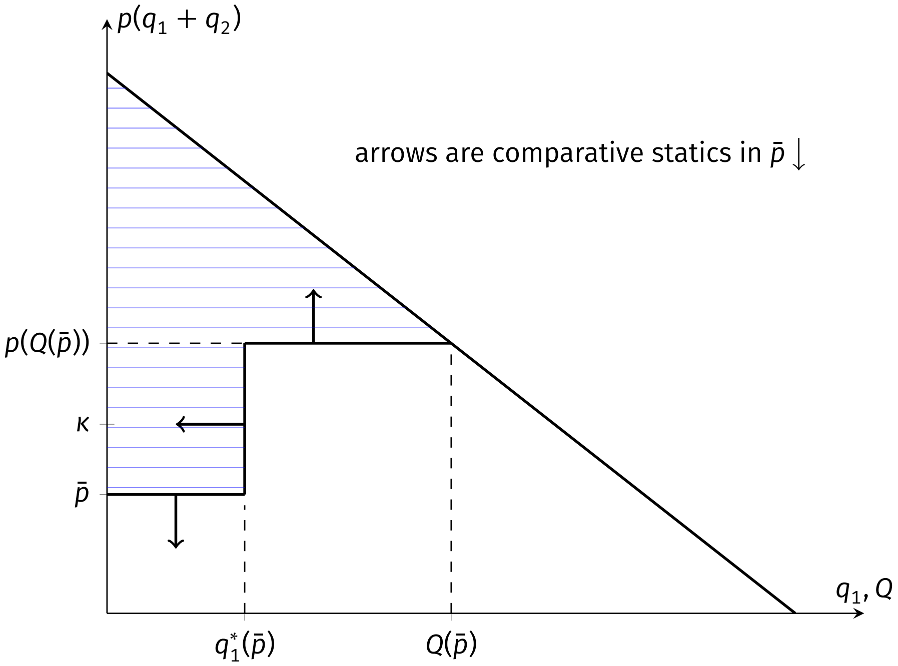


**Figure 11.** The consumer surplus is the area below the inverse demand curve less the consumers' expenditure. I replaced the equilibrium total quantity, $q_1^*(\bar{p}) + q_2^*(\bar{p})$, by $Q(\bar{p})$. Marginally decreasing the price cap has three effects on the consumer surplus. Each of them is illustrated with an arrow.

Formally, the consumer surplus in a rationing equilibrium is

$$CS(\bar{p}) = \int_0^{q_1^*(\bar{p})+q_2^*(\bar{p})} p(x)\ dx - q_1^*(\bar{p}) \cdot \bar{p} - q_2^*(\bar{p}) \cdot p(q_1^*(\bar{p}) + q_2^*(\bar{p})); \tag{21}$$

the area under the inverse demand curve up to the total quantity less the expenditure for the quantity of firm 1 less the expenditure for the quantity of firm 2.

A marginal decrease in the price cap has three effects on the consumer surplus:

$$-\frac{\partial CS(\bar{p})}{\partial \bar{p}} = q_1^*(\bar{p}) - \frac{\partial q_1^*(\bar{p})}{\partial \bar{p}} \cdot \Big(p(Q(\bar{p})) - \bar{p}\Big) - q_2^*(\bar{p}) \cdot (-p'(Q(\bar{p}))) \cdot \frac{\partial Q(\bar{p})}{\partial \bar{p}}, \tag{22}$$

where I replaced the equilibrium total quantity, $q_1^*(\bar{p}) + q_2^*(\bar{p})$, by $Q(\bar{p})$ to fit the expression in one line.

The first term in equation (22) is the gain of those consumers that buy from firm 1 before and after the marginal decrease of the price cap and save one marginal unit. In Figure 11, this is the blue rectangle's expanding down. The second term in equation (22) is the loss of those consumers that buy from firm 1 before but have to buy from firm 2 at a higher price after the marginal decrease of the price cap because firm 1 reduces its quantity. In Figure 11, this is the blue rectangle's shrinking to the left.

The third term in equation (22) is the loss of those consumers that buy from firm 2 before and after the marginal change because the price they have to pay increases as the total quantity decreases. In Figure 11, this is the blue triangle's shrinking.

The total effect of a marginal decrease in the price cap on the consumer surplus is ambiguous. The positive effect is large when $q_1^*(\bar{p})$ is large because this increases the transfer from firm 1 to the consumers. This is the case close to the cutoff, $\kappa$. The negative effects are small when the firms' quantities adjust only little because few consumers switch from firm 1 to firm 2 and the price of firm 2 rises only little. This is the case when the marginal cost is very steep at firm 1's optimal quantity.

The levels of the consumer surplus at the extreme price caps can be compared to the benchmark level. At the cutoff, the consumer surplus is larger than in the Cournot-Nash equilibrium because it has been increasing through all clearing equilibria. If the price cap goes to the marginal cost of the first unit, the consumer surplus is lower than in the Cournot-Nash equilibrium because it goes to the consumer surplus in a monopoly as firm 1 leaves the market. For the marginal change in the consumer surplus, these results imply that it has to be decreasing for at least some price caps.

In the case of a linear demand and quadratic cost (see Figure 10.), the consumer surplus increases when the price cap is decreased beginning at the cutoff. For all such functional forms, the transfer from firm 1 to the consumers outweighs the quantity reduction and price increase. Therefore, the price cap that uniquely maximizes the consumer surplus corresponds to a rationing equilibrium.

Furthermore, the consumer surplus is strictly concave within the rationing equilibria. When the price cap decreases, the positive effect gets monotonically smaller (because the quantity of firm 1 is decreasing), whereas the negative effects get monotonically larger (because the firms' reactions to a change in the price cap are constant and the price differential between the firms increases).

Whether the price cap that maximizes the consumer surplus is greater or less than the price cap that maximizes the welfare within the rationing equilibria depends on the cost parameter and on the slope of the inverse demand function. The reason is that the cost directly enters into the welfare, whereas it only enters indirectly through the price into the consumer surplus. If the cost parameter is sufficiently large compared to the slope of the inverse demand function, the consumer surplus gets maximized at a lower price cap.[15]

### 4.2 Constant Marginal Cost

When the marginal cost is a constant $c$, there is no cutoff and all equilibria are clearing equilibria. Figure 12 depicts the best response functions of both firms. Whenever firm 2's best response is positive, it is indirectly defined by the solution to its first-order condition,

$$BR_2(q_1) \;=\; q_2^*(q_1) \;:\quad p(q_1+q_2^*) + q_2^* \cdot p'(q_1+q_2^*) - c \;=\; 0. \tag{23}$$

Whenever firm 1's best response is positive, it is

$$BR_1(q_2;\bar{p}) \;=\; \max\{BR_2(q_2),\; \bar{q}-q_2\} \tag{24}$$

for reasons analogous to the case of increasing marginal cost (just let the inverse of the marginal cost function be infinity everywhere because it does not exist).

15. The exact condition is $c > (\sqrt{2}-1)\cdot b$.

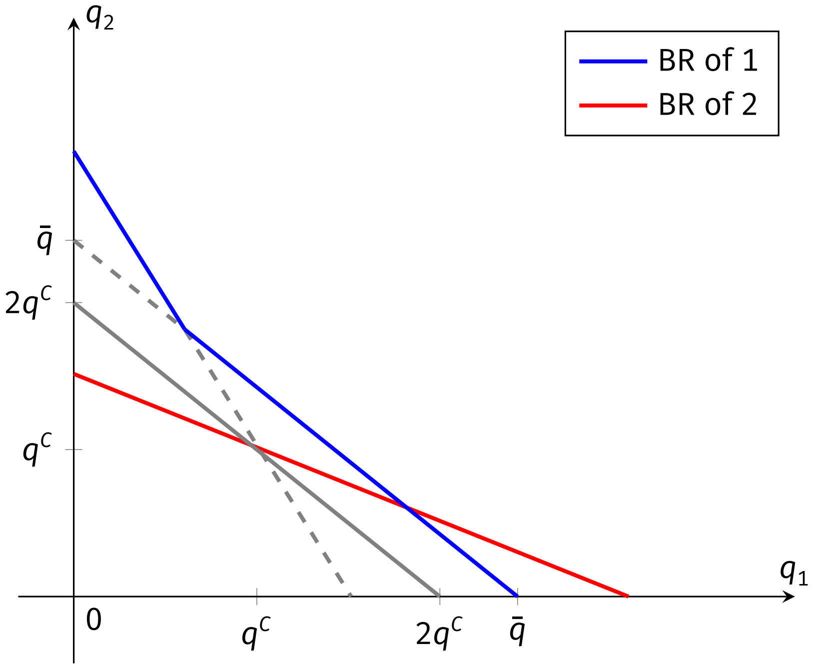


**Figure 12.** On the $x$-axis is the quantity of firm 1 and on the $y$-axis is the quantity of firm 2. The best response function of firm 2 is in red. The best response function of firm 1 is in blue. The dashed gray lines are the two parts of firm 1's best response function where they are not optimal.

**Proposition 3.** *When the marginal cost is constant at c, the unique equilibrium is*

$$q_1^*(\bar{q}) = \bar{q} - q_2^*(\bar{q}) \quad \text{and} \quad q_2^*(\bar{q}) = \frac{p(\bar{q}) - c}{-p'(\bar{q})}. \tag{25}$$

*Proof.* The proof is in Appendix A, Subsection A.6. □

### 4.3 Stackelberg Competition

This subsection explains and exploits the relationship between asymmetric price caps and the sequential Stackelberg competition. In the Stackelberg game, the Stackelberg leader chooses its quantity, $s_1$, first. The Stackelberg follower, then observes the choice of the leader and optimally chooses its own quantity, $s_2$. This allows the Stackelberg leader to choose its optimal point on the best response function of the Stackelberg follower, which is that of a standard Cournot duopolist.

Commitment power is equivalent to sequential choice: If firm 1 has the possibility to commit itself to a quantity choice, firm 1 chooses the same quantity as a Stackelberg leader does in the Stackelberg equilibrium. Firm 2 anticipates this and replies as a Stackelberg follower does in the Stackelberg equilibrium.

Because of strategic substitutability, the Stackelberg leader can profitably deviate from the Cournot-Nash equilibrium of the simultaneous game. If the Stackelberg leader increases its quantity, the Stackelberg follower reacts by reducing its quantity, which counteracts the price depressing effect of increasing the quantity in the first place: The Stackelberg follower's reaction increases the Stackelberg follower's marginal profit.

Formally, the Stackelberg leader's marginal profit, anticipating the Stackelberg follower's reaction, is

$$\frac{\partial \pi(s_1, s_2(s_1))}{\partial s_1} = p(s_1 + s_2(s_1)) - c'(s_1) + s_1 \cdot p'(s_1 + s_2(s_1)) \cdot \left(1 + \frac{\partial s_2(s_1)}{\partial s_1}\right). \tag{26}$$

Evaluated at the Cournot-Nash equilibrium quantities, it is strictly positive,

$$\begin{aligned} \left.\frac{\partial \pi(s_1, s_2(s_1))}{\partial s_1}\right|_{s_1=q^C=s_2} &= p(2q^C) - c'(q^C) + q^C \cdot p'(2q^C) \cdot \left(1 + \left.\frac{\partial s_2(s_1)}{\partial s_1}\right|_{s_1=q^C}\right) \\ &= \underbrace{q^C}_{>0} \cdot \underbrace{p'(2q^C)}_{<0} \cdot \underbrace{\left(\left.\frac{\partial s_2(s_1)}{\partial s_1}\right|_{s_1=q^C}\right)}_{<0} > 0, \end{aligned} \tag{27}$$

where the last equality follows from the Cournot-Nash equilibrium condition. Strategic substitutability implies that $\frac{\partial s_2(s_1)}{\partial s_1} < 0$.

There is a relationship between the well-known profit function of the Stackelberg leader and firm 1's equilibrium profit for different price caps. If the Stackelberg leader and firm 1 choose the same quantity, so do the Stackelberg follower and firm 2 because their best response functions are the same. So, firm 1's equilibrium profit in a clearing equilibrium is the same as the Stackelberg leader's profit when it chooses firm 1's equilibrium quantity. To get firm 1's equilibrium profit in a rationing equilibrium, the additional transfer from firm 1 to the consumers (see Subsection 4.1) has to be subtracted from the Stackelberg leader's profit. The reason is that in rationing equilibria, firm 1 has a different price—the price cap—than the Stackelberg leader when it chooses firm 1's equilibrium quantity—the price that firm 2 gets in the rationing equilibrium.

Although firm 1's price is capped, it can actually profit from having a price cap. This result follows from the relationship with the Stackelberg leader's profit function and from the monotonicity of firm 1's equilibrium quantity in the price cap. When the price cap is the Cournot-Nash price, firm 1's equilibrium quantity is the Cournot-Nash quantity. Marginally decreasing the price cap increases firm 1's equilibrium quantity. Because the Stackelberg leader's profit increases when marginally increasing its quantity starting at the Cournot-Nash quantity, so does firm 1's equilibrium profit. Thus, firm 1's equilibrium profit with a price cap marginally below the Cournot-Nash price exceeds the benchmark profit without a price cap. The reason is the strategic effect—the crowding out—that firm 1's price cap exerts on the other firm.

Assuming additionally that the Stackelberg leader's profit is strictly quasi-concave in its quantity makes firm 1's equilibrium profit strictly quasi-concave in the price cap for the clearing equilibria. As is generally true, if the price cap decreases starting from the Cournot-Nash price, the profit initially increases. Because of strict quasi-concavity, the profit keeps increasing up to its maximum at the unique Stackelberg equilibrium price. As Proposition 2 has shown, the Stackelberg equilibrium price lies within the range of clearing equilibria. If the price cap decreases further, firm 1's equilibrium profit decreases again.

In the rationing equilibria, firm 1's equilibrium profit decreases monotonically in a decreasing price cap: Both firm 1's quantity and the price it receives decrease.

Figure 13 illustrates firm 1's equilibrium profits for different price caps in the special case of linear demand and quadratic cost. For these functional forms, the Stackelberg leader's profit function is strictly quasi-concave in its quantity, so the equilibrium profit of firm 1 increases until the price cap is the Stackelberg equilibrium price and then decreases. Below the cutoff, it always decreases.

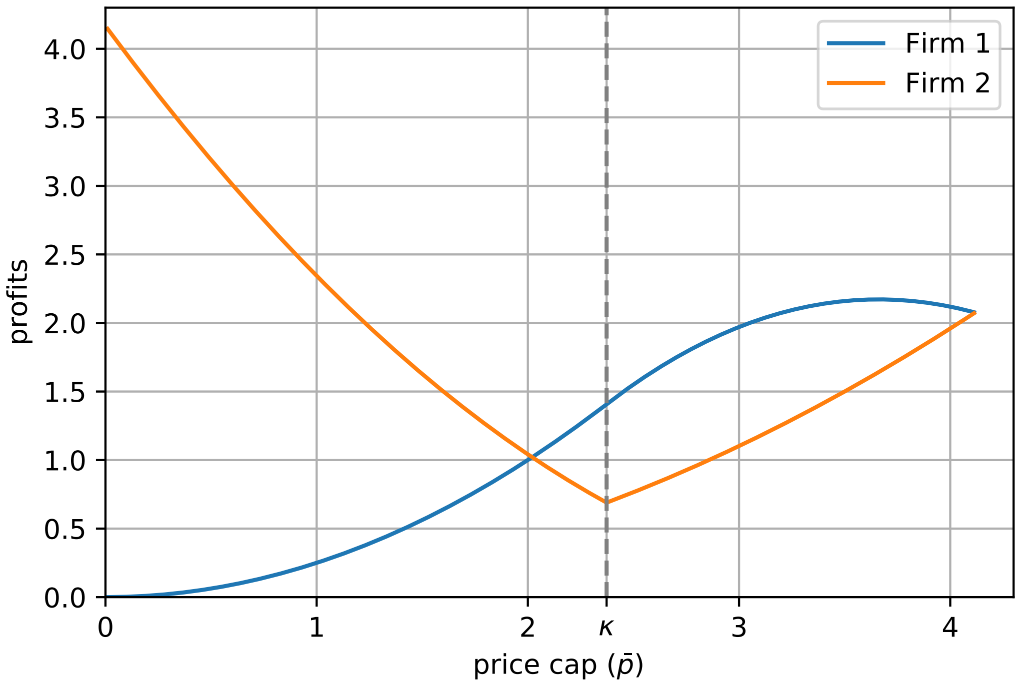


**Figure 13.** The equilibrium profits of the firms when the cost functions are $c(q_i) = q_i^2$ and the inverse demand is $p(q) = 10 - 5 \cdot q$.

The relationship to the game with asymmetric price caps offers a novel interpretation for the Stackelberg game: In the first stage, the Stackelberg leader commits to an individual price cap.[16] In the second stage, both firms choose quantities simultaneously.

The subgame perfect equilibrium in this alternative interpretation can be found by backward induction. In the second stage, given a price cap for the Stackelberg leader, the firms play the unique equilibria analyzed in the main part. In the first stage, the Stackelberg leader optimally chooses a Stackelberg equilibrium price as its price cap. The equilibrium outcome is identical to the outcome in the traditional Stackelberg game. Committing to a price cap can replace the commitment to a quantity because it eliminates the inframarginal loss in the second stage, so choosing a larger quantity becomes optimal for firm 1.

### 4.4 Both Firms Have Price Caps

This subsection deals with the case in which both firms have price caps. Without loss of generality, assume that firm 2 has a strictly higher price cap.

16. It is not necessary to set a price cap for the Stackelberg follower, too. In fact, if the Stackelberg leader could only choose a symmetric price cap for both firms, it might hurt itself. With symmetric price caps, there is a continuum of equilibria, that includes the reversed quantities (Okumura, 2017). So if the Stackelberg leader chooses the Stackelberg equilibrium price as a price cap, there is an equilibrium in which it makes the Stackelberg follower's profit.

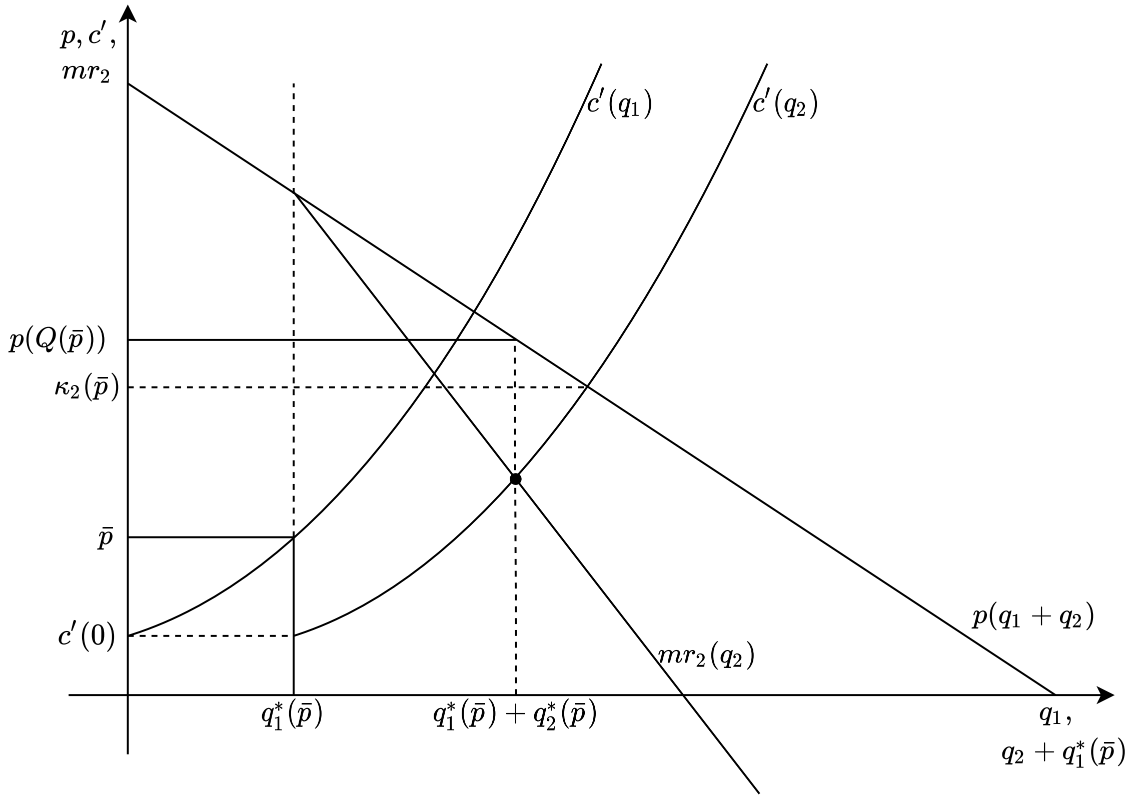


**Figure 14.** Beginning on the left, firm 1 produces the quantity at which its price cap $\bar{p}$ and its marginal cost intersect, $q_1^*(\bar{p})$. At this quantity, $q_2$ starts at 0. Firm 2's inverse residual demand curve is the original inverse demand curve from hereon (starting to count $q_2$ at $q_1^*(\bar{p})$ is the same as shifting the original inverse demand curve to the left by $q_1^*(\bar{p})$). Firm 2 produces the quantity at which its marginal revenue and its marginal cost intersect, $q_2^*(\bar{q})$. The corresponding market-clearing price, $p(q_1^*(\bar{p})+q_2^*(\bar{p}))$, is abbreviated with $p(Q(\bar{p}))$. The price at which firm 2's marginal cost and the inverse residual demand curve intersect is $\kappa_2(\bar{p})$. If firm 2's price cap, $\bar{p}_2$, is above $p(Q(\bar{p}))$, it does not bind and is ineffective. If $p(Q(\bar{p})) > \bar{p}_2 \geq \kappa_2(\bar{p})$, there is a partially rationing equilibrium. If $\bar{p} < \kappa_2(\bar{p})$, there is a doubly rationing equilibrium.

Firm 2's price cap might affect its own equilibrium quantity, but never firm 1's equilibrium quantity. If firm 1's price cap is above the cutoff, $\kappa$, there is a clearing equilibrium and firm 2's price cap does not bind, so it neither affects firm 1's nor its own equilibrium quantity. If firm 1's price cap is below the cutoff, $\kappa$, there is a rationing equilibrium. Because firm 1's price cap strictly binds, it plays the price-takers quantity, $(c')^{-1}(\bar{p})$ in any equilibrium.

Whether firm 2's price cap affects its own quantity in a rationing equilibrium depends on its level. As firm 2 is the monopolist on the market for the residual demand, there are the same three cases as for the monopolist in Appendix B: If the price cap is above the monopoly price, $p(q_1^*(\bar{p})+q_2^*(\bar{p}))$, it has no effect; if the price cap is between the monopoly price and the competitive price on the market for residual demand, $\kappa_2(\bar{p})$, it binds and the price clears the market; if the price cap is between the marginal cost for the first unit and the competitive price, the price cap binds and the price does not clear the market.

**Proposition 4.** *Assume that firm 1 has the price cap $\bar{p}$ and firm 2 has a price cap $\bar{p}_2$. Without loss of generality, assume that $\bar{p} < \bar{p}_2$.*

*Define $\kappa_2(\bar{p})$ as the value at which $p(q_1^*(\bar{p})+q_2)$ and $c'(q_2)$—both functions of $q_2$—intersect; illustrated in Figure 14. $\kappa_2(\bar{p})$ is decreasing in $\bar{p}$.*

*(i) If $\bar{p} \geq \kappa$, the only pure-strategy Nash equilibrium is the clearing equilibrium described in Theorem 1. It does not depend on $\bar{p}_2$.*

*(ii) If $\bar{p} < \kappa$ and $\bar{p}_2 \geq p(q_1^*(\bar{p})+q_2^*(\bar{p}))$ (as defined in Theorem 2), the only pure-strategy Nash equilibrium is the rationing equilibrium described in Theorem 2. It does not depend on $\bar{p}_2$.*

*(iii) If $\bar{p} < \kappa$ and $p(q_1^*(\bar{p})+q_2^*(\bar{p})) > \bar{p}_2 \geq \kappa_2(\bar{p})$, the only pure-strategy Nash equilibrium is a partially rationing equilibrium. In this equilibrium, firm 1 produces $q_1^*(\bar{p}) = (c')^{-1}(\bar{p})$ and firm 2 produces the quantity that brings the market-clearing price to $\bar{p}_2$, which is $q_2^*(\bar{p},\bar{p}_2) = p^{-1}(\bar{p}_2) - q_1^*(\bar{p})$.*

*(iv) If $\bar{p} < \kappa$ and $\kappa_2(\bar{p}) > \bar{p}_2$, the only pure-strategy Nash equilibrium is a doubly rationing equilibrium. In this equilibrium, firm 1 produces $q_1^*(\bar{p}) = (c')^{-1}(\bar{p})$ and firm 2 produces $q_2^*(\bar{p}_2) = (c')^{-1}(\bar{p}_2)$, which depends only on the own price cap.*

*Proof.* The proof is in Appendix A, Subsection A.7. □

When the marginal cost is constant, neither the existence nor the level of the higher price cap influence the equilibrium quantities, as Proposition 5 shows. The reason is that there are only clearing equilibria in which firm 2's price cap cannot bind.

**Proposition 5.** *Assume that the marginal cost is constant. Assume that firm 1 has the price cap $\bar{p}$ and firm 2 has a price cap $\bar{p}_2$ with $\bar{p} < \bar{p}_2$. Then, for all $\bar{p} \in (c, p^C)$, the only pure-strategy Nash equilibrium is the clearing equilibrium described in Proposition 3.*

*Proof.* The arguments in the proof of Proposition 3 are still true: There can be no equilibrium with $q_1 + q_2 < \bar{q}$ because firm 1 could profitably deviate, and there can be no equilibrium with $q_1 + q_2 > \bar{q}$ because at least one firm could profitably deviate.

So, if there is an equilibrium, it has to be that $q_1 + q_2 = \bar{q}$. But if the total quantity is $\bar{q}$, the price cap of firm 2 does not bind and it has, thus, no effect. □

These results are extreme, especially for clearing equilibria. The continuum of equilibria in the case of symmetric price caps described in Okumura (2017) collapses to one of its boundaries if there is the slightest asymmetry in the price caps. The welfare implication of this result is that the inefficiency on the production side arises discontinuously when one of two perfectly symmetric price caps is changed marginally.

The existence of the continuum of equilibria hinges on the fact that both firms' price caps bind simultaneously, so both firms are at the drop in their marginal profits. This creates some leeway in satisfying the optimality conditions. Asymmetric price caps cannot, however, bind simultaneously. So, the firm with the non-binding price cap has a unique best response, pinning down the best response of the other firm.

### 4.5 Heterogeneous Cost Functions

In the main part, I assume that the firms have the same cost function. This assumption makes the interpretation of the price cap's welfare effects simpler, but it is, for example, not necessary to determine the signs of the comparative statics in the price cap.

Heterogeneous cost functions do not change the comparative statics of the equilibrium quantities in the price cap. As mentioned above, the comparative statics follow from the assumption that the inverse demand function is strictly log-concave, implying strategic substitutability. More accurately, the comparative statics depend on the slope of firm 2's best response function's being strictly between -1 and 0, which is implied by the strict log-concavity of the residual inverse demand function and the marginal cost's being strictly increasing (see the proof of Proposition 1). Neither the strict log-concavity of firm 2's residual inverse demand function nor the monotonicity of its marginal cost needs homogeneous costs. In fact, the slope of firm 2's best response function does not depend on firm 1's cost function at all.

To see how heterogeneous cost functions change the equilibrium quantities, it is helpful to look at the best response functions in a simple example. Let firm 1 have higher marginal cost for each quantity: The marginal cost of firm 1 is $\alpha \cdot c'(q_1)$, with $\alpha > 1$, whereas the marginal cost of firm 2 is $c'(q_2)$. Figure 15 illustrates how the best response function of firm 1 differs from the case with homogeneous cost. The first of the three parts is the best response of a standard Cournot duopolist, so higher marginal costs lead to smaller best responses. Because there can still be no equilibrium in the first part, this does not matter.[17] The second part, $\bar{q} - q_2$, is the quantity at which the price cap just binds, so it is independent of the cost function. The third, vertical part, $(c')^{-1}(\bar{p})$ is smaller when the marginal cost is higher.

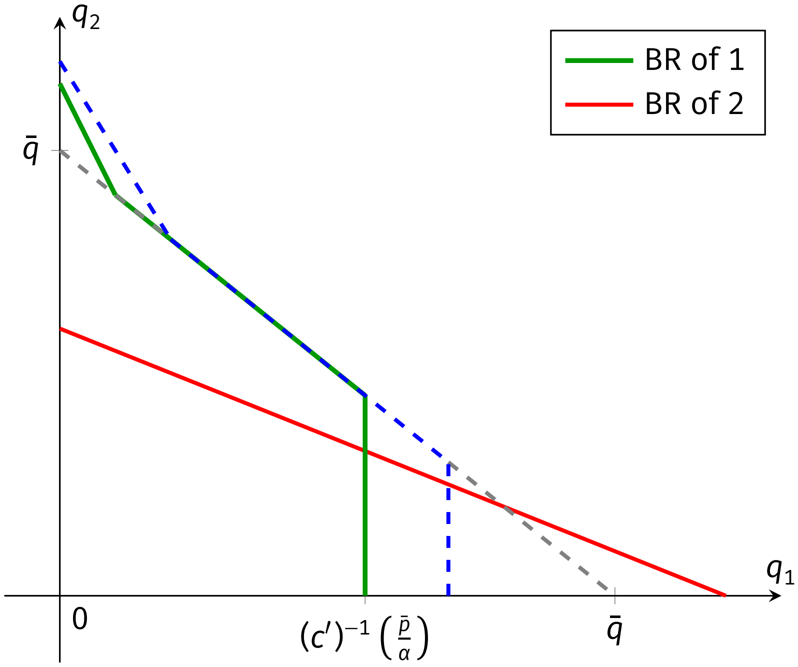


**Figure 15.** The red curve is the best response function of firm 2. The blue, dashed curve would be the best response function of firm 1 if it had the same cost function as firm 2. The green curve is the actual best response function of firm 1 when its marginal cost is $\alpha \cdot c'(q_1)$, with $\alpha > 1$, whereas the marginal cost of firm 2 is $c'(q_2)$.

As the second part of firm 1's best response function is unchanged, so are the equilibrium quantities in clearing equilibria. The cutoff between clearing and rationing equilibria, however, is a larger price cap: The marginal profit above the drop hits zero at a higher price cap because the higher marginal cost reduces the marginal profit. In rationing equilibria, firm 1's equilibrium quantity is smaller and so firm 2's equilibrium quantity, its best response, is larger due to strategic substitutability.

17. Proposition 1 follows from the fact that the slopes of both best response functions are strictly between -1 and 0.

The trade-off between total quantity and production efficiency remains. Although firm 1 produces less than firm 2 in the benchmark equilibrium without price caps, its marginal cost is larger.[18] If a price cap above the cutoff is introduced and marginally decreased, firm 1's equilibrium quantity increases—the comparative statics having the same sign as with homogeneous cost. Thus, firm 1 produces more and its expensive production crowds out the socially cheaper production of firm 2.

If, on the other hand, firm 2 has higher marginal costs than firm 1, the trade-off vanishes for at least some price caps. If firm 2's marginal cost function is $\alpha \cdot c'(q_2)$, firm 2's best response is lower than in the symmetric case for all $q_1$. In the benchmark equilibrium without price caps, it is now firm 2 that has the higher marginal cost. If firm 1 gets a price cap marginally below the Cournot-Nash price, firm 1 produces more, crowding out some of firm 2's expensive production. Thus, starting in any clearing equilibrium in which firm 2 has the higher marginal cost, marginally decreasing the price cap not only increases the total quantity but also makes the production more efficient. So, lower price caps unambiguously improve the welfare. Due to the symmetry between clearing and rationing equilibria, this is also true for higher price caps in the corresponding rationing equilibria.

### 4.6 Mixed-Strategy Nash Equilibria

In the main part, I restrict the analysis to pure-strategy Nash equilibria. I cannot rule out that mixed-strategy Nash equilibria exist. There is, however, an alternative assumption on the primitives implying strategic substitutability under which all results remain true and under which no mixed-strategy Nash equilibrium exists.

Keep all assumptions in this chapter except for Assumption 1 that the inverse demand function is strictly log-concave. Replace Assumption 1 with Assumptions 2 and 3 of Theorem 3 in Novshek (1985, p. 90):[19]

$$\exists Z \; : \; p(Z) \; = \; 0 \tag{30}$$
$$\forall q \; \in \; [0, Z) \; : \; p'(q) + q \cdot p''(q) \; < \; 0. \tag{31}$$

These assumptions imply that, without price caps, each firm's marginal revenue is weakly decreasing in the other firm's quantity and, thus, also in the own quantity. Combined with strictly increasing marginal cost, the assumptions imply that each firms' profit function is strictly concave in the own quantity.

In the game with price caps, firm 2's profit function is the same as that of a standard Cournot duopolist, so it is still strictly concave in $q_2$ for all $q_1$. Then, firm 2's expected profit, facing a mixed

18. To see this, look at the optimality conditions

$$\begin{aligned} p(q_1^* + q_2^*) + q_1^* \cdot p'(q_1^* + q_2^*) - \alpha \cdot c'(q_1^*) \; &= \; 0 \\ p(q_1^* + q_2^*) + q_2^* \cdot p'(q_1^* + q_2^*) - \phantom{\alpha \cdot} c'(q_2^*) \; &= \; 0. \end{aligned} \tag{28}$$

Because the marginal cost is strictly increasing and $\alpha > 1$, it follows that $q_1^* < q_2^*$. Then,

$$p(q_1^* + q_2^*) + q_1^* \cdot p'(q_1^* + q_2^*) \; > \; p(q_1^* + q_2^*) + q_2^* \cdot p'(q_1^* + q_2^*). \tag{29}$$

Thus, $\alpha \cdot c'(q_1^*) > c'(q_2^*)$.

19. Curiously, the alternative assumptions from Novshek (1985) are neither stronger nor weaker than the original assumption (which is the assumption from Amir, 1996). I have chosen the original assumption because the range on which the assumption holds could be constrained such that it is weaker than the alternative assumption while all of my results remain true (I have not actually constrained the range in Assumption 1 to keep the exposition as simple as possible). For a discussion about the relationship between the two alternative assumptions and how to constrain the range of Assumption 1, see Amir (2005).

strategy of firm 1, is also strictly concave. Thus, its best response is unique. So, firm 2 has to play a pure strategy in each Nash equilibrium. The best response of firm 1 to a pure strategy is also a pure strategy, as shown above. Thus, in any Nash equilibrium, both firms play pure strategies.

### 4.7 Proportional Rationing

As explained in Section 2, asymmetric price caps make it necessary to assume a rationing rule to determine the inverse residual demand function. In the main part, I have considered the efficient rationing rule, meaning that the consumer surplus is always maximized. The other common assumption in the literature is proportional rationing. With proportional rationing, the firms serve all consumers that want to buy at a price with equal probability.

To understand what this means, it is helpful to consider a microfoundation of a demand function: There is a continuum of consumers with unit demand and each consumer's valuation (their willingness to pay) for the good is drawn according to a distribution.[20] The demand function states for each price the measure of people with valuations exceeding this price—who are willing to buy at that price.

Proportional rationing includes an implicit assumption about the timing of purchasing actions: If there is excess demand at the price cap, everyone tries to buy at the cheaper price from firm 1 first.[21] The quantity of firm 1 is then allocated by proportional rationing. Consumers that get served leave the market.

Firm 2's residual demand consists of those consumers that did not get served by firm 1 and remain on the market. If there is no excess demand for the quantity of firm 1, $q_1 \geq \bar{q}$, there is no rationing and all consumers with valuations above $p(q_1)$ get served and leave the market. Firm 2's residual demand is then described by $p(q_1 + q_2)$ as in the standard Cournot case. If there is excess demand for the quantity of firm 1, $q_1 < \bar{q}$, each consumer with a valuation of at least $\bar{p}$ has the same probability of getting served by firm 1 and leaving the market, $\frac{q_1}{\bar{q}} < 1$. The counter probability of not getting served and remaining on the market is $\frac{\bar{q}-q_1}{\bar{q}}$. The densities of the remaining consumers' valuations consist of two parts. Up to the price cap of firm 1, the measures of consumers in firm 2's residual demand are given by the initial density—as none of these consumers was served. For the prices above the price cap, the initial density is multiplied with the probability of remaining on the market. The construction of the density of the remaining consumers' valuations is illustrated in Figure 16.

The inverse residual demand function of firm 2 depends on whether firm 1's price cap binds or not. If $q_1 + q_2 < \bar{q}$, the price cap binds and firm 2 sells only to those who participated in the lottery for the quantity of firm 1 but lost. When deriving the inverse demand function from the distribution of valuations of the consumers, the axis are inverted, so firm 2's inverse residual demand function is $p\left(\frac{\bar{q}}{\bar{q}-q_1} \cdot q_2\right)$. If $q_1 + q_2 \geq \bar{q}$, the price cap does not bind. As the price clears the market, the inverse residual demand function of firm 2 is given by $p(q_1 + q_2)$. To sum up, the inverse residual demand function of firm 2 is

$$p_2(q_1, \bar{p}, q_2) = \begin{cases} p\left(\frac{\bar{q}}{\bar{q}-q_1} \cdot q_2\right) & \text{if } q_2 < \bar{q} - q_1 \\ p(q_1 + q_2) & \text{if } q_2 \geq \bar{q} - q_1. \end{cases} \tag{32}$$

20. The continuum prevents market power problems, justifies considering goods to be divisible, and implies that the demand is not stochastic.

21. With efficient rationing, the timing does not matter. Firm 2 knows that in the end, all production will end up with the measure $q_1 + q_2$ of consumers with the highest willingness to pay, so the last served consumer pins down its price at $p(q_1 + q_2)$.

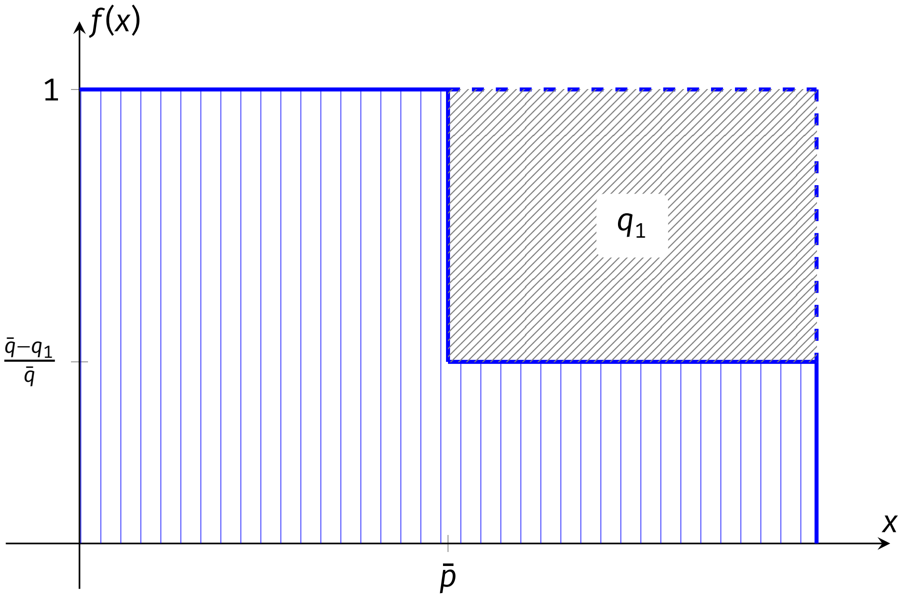


**Figure 16.** On the $x$-axis is the valuation, on the $y$-axis is the density. For simplicity, I chose a uniform distribution, which yields a linear demand function. Above $\bar{p}$, some consumers got served and left the market. The measure of consumers that got served is the gray area of size $q_1$. The blue line is the density of the remaining consumers' valuations, from which the residual demand for firm 2 is derived.

Intuitively, the inverse residual demand function is a compression in the quantity direction of the original inverse demand function for $q_2 < \bar{q} - q_1$ as some consumers with a high valuation have already been served by firm 1. The inverse residual demand and how it can be constructed graphically from the original inverse demand, firm 1's quantity, and the price cap are illustrated in Figure 17.

Firm 2's profit function consists of two functions,

$$\pi_2(q_1, \bar{p}, q_2) = \begin{cases} q_2 \cdot p\left(\frac{\bar{q}}{\bar{q}-q_1} \cdot q_2\right) - c(q_2) & \text{if } q_2 < \bar{q} - q_1 \\ q_2 \cdot p(q_1 + q_2) - c(q_2) & \text{if } q_2 \geq \bar{q} - q_1. \end{cases} \tag{33}$$

Both functions are individually strictly quasi-concave in $q_2$ because the corresponding inverse residual demand functions are strictly log-concave in $q_2$. Thus, the profit-maximizing quantity of each individual function is determined by the intersection of the marginal profit and zero.

Determining the profit-maximizing quantity of the actual profit function is, however, complicated. The marginal profit jumps upwards at $q_2 = \bar{q} - q_1$. The left-derivative and at the right-derivative at $q_2 = \bar{q} - q_1$ are

$$\begin{aligned} \left.\frac{\partial_- \pi_2(q_1, \bar{p}, q_2)}{\partial q_2}\right|_{q_2 = \bar{q} - q_1} &= \bar{p} + (\bar{q} - q_1) \cdot \frac{\bar{q}}{\bar{q} - q_1} \cdot p'(\bar{q}) - c'(\bar{q} - q_1) \quad < \\ \left.\frac{\partial_+ \pi_2(q_1, \bar{p}, q_2)}{\partial q_2}\right|_{q_2 = \bar{q} - q_1} &= \bar{p} + (\bar{q} - q_1) \cdot \qquad\quad p'(\bar{q}) - c'(\bar{q} - q_1). \end{aligned} \tag{34}$$

The inequality follows from $\frac{\bar{q}}{\bar{q}-q_1} > 1$. As some consumers with valuations above $\bar{p}$ have left the market, firm 2 depresses its price more by increasing its quantity, so the inframarginal loss is larger.

Thus, there are three cases for the profit-maximizing quantity of firm 2, depending on the sign of its marginal profit before and after the jump at $\bar{q} - q_1$. If the marginal profit begins in the negative and is still negative after the jump, the profit-maximizing quantity is given by the profit-maximizing quantity

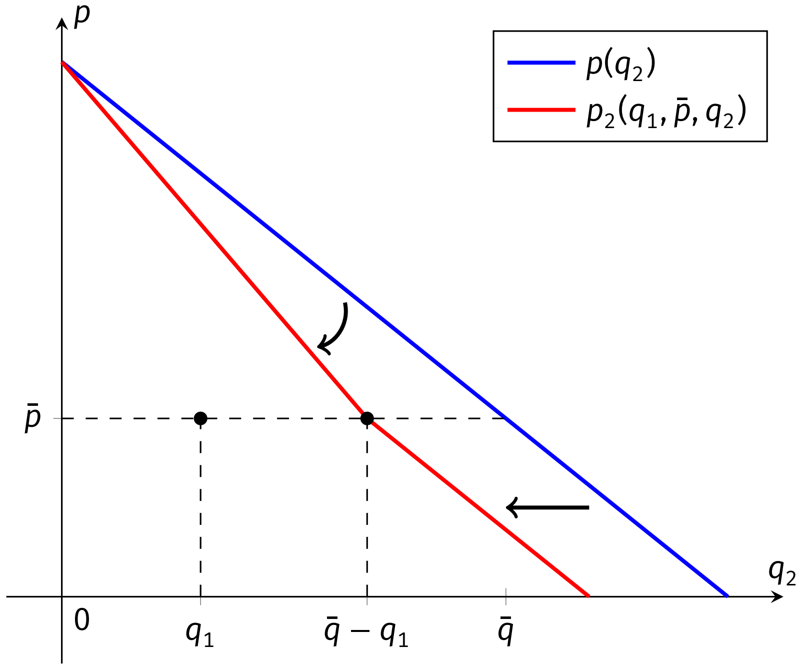


**Figure 17.** The inverse demand function is in blue. Firm 1 has the price cap $\bar{p}$ and chooses the quantity $q_1 < \bar{q}$. Firm 2's inverse residual demand function is in red. Consumers with a valuation larger than $\bar{p}$ are served with probability $\frac{q_1}{\bar{q}}$, so firm 2's inverse residual demand curve is a compression in the $x$-direction of the inverse demand curve for $q_2 < \bar{q} - q_1$. For a linear inverse demand curve, compressing is the same as tilting inwards. For $q_2 \geq \bar{q} - q_1$, the inverse demand curve is shifted to the left as in the standard Cournot case to get the inverse residual demand curve.

of the function that is the first part of firm 2's profit function. If the marginal profit begins in the positive, then it is still positive after the jump, and the profit-maximizing quantity is given by the profit-maximizing quantity of the function that is the second part of firm 2's profit function. If, however, the marginal profit begins in the negative and is positive after the jump, then either of the profit-maximizing quantities in the two parts may be the profit-maximizing quantity: Firm 2's profit function is not strictly quasi-concave in $q_2$.

As a consequence of firm 2's marginal profit's jumping up, clearing equilibria do not exist (see Proposition 6). The marginal profit jumps at exactly the quantity that makes firm 1's price cap just bind. If firm 2 provides less quantity, there is proportional rationing, so firm 2 has a stronger effect on its price. If firm 2 provides more quantity, there is price rationing, so firm 2 has a weaker effect on its price. For these reasons, it is optimal for firm 2 to either provide less quantity and to profit from a higher price or to expand its quantity further and to profit from depressing its price less. Thus, it is never a best response of firm 2 to bring the total quantity to $\bar{q}$ and to make firm 1's price cap just bind.

Moreover, as the fact that firm 2's best response function jumps over the middle part of firm 1's best response function suggests, there is no equilibrium at all for some price caps below the Cournot-Nash price. Proposition 6 shows that firm 1's price cap has to bind in all equilibria; otherwise at least one firm could profitably deviate as both firms became standard Cournot duopolists. Thus, in any equilibrium, the total quantity is either $\bar{q}$ (so the equilibrium would be clearing) or less (so the equilibrium would be rationing). Therefore, to show that there is no equilibrium, it is sufficient to verify that there is no rationing equilibrium for a given price cap. Intuitively, firm 2 must discretely decide whether to stay on its small isolated market and to produce a small quantity such that there is rationing or to produce a large quantity such that both firms sell on the same market without a price cap or rationing. Due to the jump in its marginal profit, the latter would mean a total quantity exceeding $\bar{q}$. Numerical exercises verify that for high price caps, given the rationing equilibrium quantity of firm 1, it is optimal for firm 2 to produce a large quantity: There is no equilibrium.

Rationing equilibria exist when it is optimal for firm 2 to stay on its isolated small market, as this means that firm 1's price cap strictly binds and firm 1 acts as a price-taker. With proportional rationing, the cutoff below which there are rationing equilibria is larger as Proposition 6 shows. The comparative statics, have the same signs: If the price cap falls, firm 1 produces less and firm 2 produces more.

The trade-off between a larger total quantity and the production efficiency, however, might vanish. With proportional rationing, the slope of firm 2's best response function is not necessarily bounded between 0 and $-1$. Thus, when the price cap falls and firm 1 produces less, it might be that firm 2 expands it quantity sufficiently to increase the total quantity. The reason is an additional positive effect on firm 2's quantity when the price cap decreases: More consumers with high valuations remain on the market as firm 1 reduces its supply and more consumers with lower valuations enter the lottery for the quantity of firm 1. Thus, firm 2's residual demand gets less compressed and firm 2's depressing effect on its price gets attenuated.

There is, however, an additional welfare-reducing effect from misallocation on the consumers' side of the market. For fixed quantities, a lower price cap leads to more misallocation because more consumers with a low valuation enter and win the lottery for the quantity of firm 1. For adjusting quantities in rationing equilibria, the effect is ambiguous: While more consumers with low valuations enter the lottery, firm 2 replaces some of the quantity of firm 1, and the quantity of firm 2 is allocated efficiently.

**Proposition 6.**

*(i) There are no equilibria with $q_1 + q_2 > \bar{q}$ (Proposition 1 remains true with proportional rationing).*

*(ii) There are no clearing equilibria with $q_1 + q_2 = \bar{q}$.*

*(iii) There are rationing equilibria with $q_1 + q_2 < \bar{q}$. There is an $\epsilon > 0$, such that the unique equilibrium for each price cap in $(c'(0), \kappa + \epsilon)$ is a rationing equilibrium. When the price cap decreases, firm 1 decreases and firm 2 increases its quantity.*

*Proof.* The proof is in Appendix A, Subsection A.8. □

### 4.8 Symmetric Regulation of Vertically Differentiated Products

Often, price cap regulation is symmetric for all firms. If the products of the firms are, however, vertically differentiated, symmetric regulation is in fact asymmetric, as the following example shows.

**Example 1.** In the initial situation, there is no price cap. The goods are vertically differentiated, with firm 1 offering the superior product. The inverse residual demand function for the worse good of firm 2 is $p_2(q_1 + q_2)$. The inverse residual demand function for the superior good of firm 1 is $p_1(q_1 + q_2) \equiv p_2(q_1 + q_2) + x$, with $x > 0$. Thus, the price for the good of firm 1 is always by $x$ units larger than the price for the good of firm 2. A possible microfoundation is that each consumer's marginal willingness to pay is larger by $x$ for good 1 than for good 2, reflecting the superiority of good 1. This form of vertical differentiation is proposed in Ritz (2018).

Now, both firms get the same price cap $\bar{p}$.

The price caps stop binding at different total quantities because the goods are differentiated. For firm 2, the price cap stops binding if the total quantity exceeds $\bar{q}_2 \equiv p_2^{-1}(\bar{p})$. Because the price for the good of firm 1 is larger for each total quantity, price cap stops binding for firm 1 at a larger quantity, $\bar{q}_1 \equiv p_2^{-1}(\bar{p} - x)$. Effectively, firm 1's price cap is tighter than firm 2's price cap.

Because the firms' price caps cannot just bind simultaneously, the situation with vertically differentiated goods is comparable to the situation with asymmetric price caps (Subsection 4.4). Symmetric price caps on vertically differentiated goods cause the same misallocation of quantities on the producers' side of the market as asymmetric price caps. There is, however, an additional benefit to the welfare because the production is distorted in favor of the superior good.

### 4.9 Applications

One application of my model is concerned with the modeling of mixed oligopolies (Cremer, Marchand, and Thisse, 1989, Fraja and Delbono, 1989, and Fraja and Delbono, 1990). In mixed oligopolies, the oligopolists have different objectives. Private firms maximize their profits and public firms maximize the welfare. A typical finding is that oligopolists trying to maximize the welfare sometimes make larger profits than profit-maximizing firms because their expanding their quantity makes other firms reduce their quantities.[22]

I propose a new approach to modeling mixed oligopolies: Letting the regulator choose a price cap for the public firm instead of choosing a quantity. Price capping public firms seems more realistic, is equally tractable, and has slightly different implications that could be used to empirically evaluate the different modeling approaches.

A first result follows from the analysis of heterogeneous cost functions (see Subsection 4.5). A common assumption in the literature on mixed oligopolies is that the public firm is less efficient than the private firm. In such settings, applying a price cap to the public firm leads to a trade-off between production efficiency and total quantity. It would be better if the regulator could apply price caps to the private firms.

My analysis, however, also shows that even if the public firm is less efficient, price-capping the public firm can improve the welfare. Future research could focus on how to identify the optimal price cap by using only observable data, such as market shares or whether a market is segmented into low-price and high-price parts.

Another application is price stickiness. In some situations, shocks leave prices unchanged, although standard arguments would predict price changes. My model is applicable to price stickiness in two ways.

On the one hand, as mentioned in the introduction, asymmetric price stickiness across firms are a microfoundation of asymmetric price caps. For example, the fairness considerations of the "invisible handshake" (Okun, 1981): Consumers become regular customers to save on search costs and the firm, in return, forgoes "unfair" price increases. Thus, when a shock happens that passing on would be considered

22. For the benevolent firm, it is of course bad that the private firms reduce their quantities as the objective is to maximize the welfare, not the profit.

unfair, a firm with a large share of regular customers would have a high cost of adjusting its price—a price cap. A firm with few regular customers would not have a high cost of adjusting its price. My model offers a framework to think about and to estimate the welfare consequences of such price stickiness.

On the other hand, my model offers a novel explanation for why firms might want to make their prices stickier to increase their profits.[23] As Subsection 4.3 has illustrated, the firm with a price cap can make larger profits than in the benchmark. By making price adjustments costly, firms can make the current Cournot-Nash price their price cap. If a small cost shock happens, increasing the Cournot-Nash price marginally, a firm that has capped its price at the old Cournot-Nash price crowds out the other firm's production and makes larger profits.

## 5 Conclusion

My innovation is adding asymmetric price caps to the canonical Cournot quantity competition model. Besides standard regularity assumptions, the asymmetric price caps make it necessary to assume a rationing rule to determine the firms' residual demands. I consider the efficient and the proportional rationing rule.

My main result is that asymmetric price caps distort the production efficiency. Moreover, in many settings, there is a trade-off between the total quantity in the market and the production efficiency. Hence, the welfare is not necessarily improved when a price cap is changed and a larger quantity is traded. Nevertheless, there are always asymmetric price caps that increase the welfare compared to the benchmark without any price caps.

My contribution to the literature is showing that with asymmetric price caps, distorted production is not a possibility, but an inevitability. In the existing literature with symmetric price caps, there is a continuum of equilibria with the same total quantities, which also contains a symmetric equilibrium with efficient production (Okumura, 2017). A tiny asymmetry in the price caps is sufficient for the continuum to collapse to a unique equilibrium at one of its boundaries, which potentially causes substantial waste in the production. An example suggests that an asymmetry in the goods might have the same effect: A setting with symmetric price caps on vertically differentiated goods can be reformulated into a setting with asymmetric price caps on identical goods.

Further, while clearing equilibria have been known, I introduce rationing equilibria. If the lower price cap is above a cutoff, the unique pure-strategy Nash equilibrium is a clearing equilibrium. Both firms' equilibrium price is the lower price cap and there is no rationing in equilibrium. Because the firm with the binding price cap does not depress its own price when expanding its quantity as long as the price cap binds, it produces a larger quantity in equilibrium. A larger production of the firm with the binding price cap crowds out production of the other firm. Whenever this means that socially expensive production crowds out socially cheap production, there is the trade-off between quantity and production efficiency. For the firms, the crowding out implies that a firm can induce the Stackelberg equilibrium by committing to a price cap instead of a quantity in a first stage.

23. One existing explanation is habit-forming goods (Nakamura and Steinsson, 2011): Consumers do not consume habit-forming goods because they anticipate being exploited afterwards—unless the firm makes its prices sticky to commit to not exploiting the consumers.

If the lower price cap is below a cutoff, the unique pure-strategy Nash equilibrium is a rationing equilibrium. The firms have different equilibrium prices and there is non-price rationing in equilibrium. The firm with the lower price cap behaves as a price-taker and produces until its marginal cost equals its price cap, at which it sells. The other firm sells at a higher price. Only in this case, it makes a difference whether the other firm has a price cap, too. Because the other firm is the monopolist on the market for residual demand, its price cap has the same effect as a price cap in a monopoly: It might be non-binding, it might bind while the price clears the market for residual demand, or it might bind while there is also non-price rationing on the market for residual demand.

Although seemingly different, both types of equilibria are symmetric to each other if the higher price cap is non-binding. Whenever the total equilibrium quantity is the same, both firms' equilibrium quantities are the same. Thus, also the welfare and the trade-off between quantity and production efficiency are the same.

In the special case with linear demand and quadratic cost, observable equilibrium outcomes predict whether price-capping only one firm is better than no regulation: Whenever the price-capped firm produces a larger quantity, the welfare is strictly larger than without a price cap, although the production is distorted. In particular, at the price cap at which the market starts to segment into a low-price and a high-price part, which might be observable, the welfare is higher than without a price cap, although it attains a local minimum.

A possible avenue for further research is adjusting the model to evaluate real-world regulation. As the example of linear demand and quadratic cost has shown, making structural assumptions can allow identifying welfare implications from observable data. Another avenue for future research is to tackle open questions. The subsection on proportional rationing has shown that it is difficult to determine equilibria with this rationing rule. As a consequence, it is also difficult to evaluate the welfare effects of asymmetric price caps, taking into account the misallocation on the consumers' side of the market. Future research could extend the results for proportional and alternative rationing rules, maybe combining results with the general approach in (Bulow and Klemperer, 2012), which takes industry supply functions as given.

# Appendix A Proofs

## A.1 Proof of Proposition 1

*If $q_1 + q_2 > \bar{q}$, then the marginal profit of at least one firm is strictly negative.*

Figure 8 illustrates the idea of the proof.

*Proof.* Because the price cap does not bind, $q_1 + q_2 > \bar{q}$, both firms' marginal profits are as in the standard Cournot model. Applying the implicit function theorem to the first-order condition yields a bound for the slope of the best response function.

Firm $i$'s first-order condition is

$$\frac{\partial \pi_i(q_i^*, q_j)}{\partial q_i} = p(q_i^* + q_j) + q_i^* \cdot p'(q_i^* + q_j) - c'(q_i^*) \overset{!}{=} 0. \tag{A.1}$$

Note that

$$\frac{\partial^2 \pi_i(q_i^*, q_j)}{\partial q_i^2} = 2 \cdot p'(q_i^* + q_j) + q_i^* \cdot p''(q_i^* + q_j) - c''(q_i^*) \tag{A.2}$$

$$= \frac{\partial^2 \pi_i(q_i^*, q_j)}{\partial q_i \, \partial q_j} + p'(q_i^* + q_j) - c''(q_i^*), \tag{A.3}$$

which is strictly negative because of strategic substitutability.

Applying the implicit function theorem to (A.1) yields

$$\frac{\partial q_i^*(q_j)}{\partial q_j} = -\frac{p'(q_i^* + q_j) + q_i^* \cdot p''(q_i^* + q_j)}{2 \cdot p'(q_i^* + q_j) + q_i^* \cdot p''(q_i^* + q_j) - c''(q_i^*)} \tag{A.4}$$

$$= -\frac{\frac{\partial^2 \pi_i(q_i^*, q_j)}{\partial q_i \, \partial q_j}}{\frac{\partial^2 \pi_i(q_i^*, q_j)}{\partial q_i^2}} = -\frac{\frac{\partial^2 \pi_i(q_i^*, q_j)}{\partial q_i \, \partial q_j}}{\frac{\partial^2 \pi_i(q_i^*, q_j)}{\partial q_i \, \partial q_j} + p'(q_i^* + q_j) - c''(q_i^*)} > -1. \tag{A.5}$$

The inequality follows because all terms are negative (the cross derivative because of strategic substitutability, and $p'$ and $-c''$ by assumption).

Because the slope of the best response function is always between 0 and -1, the total quantity cannot exceed $2q^C$ without making the marginal profit of at least one firm strictly negative. The marginal profits of both firms are 0 if both firms produce $q^C$. If now one firm produces one marginal unit more, this firm's marginal profit becomes negative unless the other firm reduces its quantity by the reciprocal of the slope of the best response function of the other firm, which means by more than one marginal unit. This holds for all marginal units. □

## A.2 Proof of Theorem 1

*The quantities*

$$q_1^*(\bar{q}) = \bar{q} - q_2^*(\bar{q}) \quad \textit{and} \quad q_2^*(\bar{q}) : p(\bar{q}) + q_2^*(\bar{q}) \cdot p'(\bar{q}) - c'(q_2^*(\bar{q})) = 0 \tag{11}$$

*have the properties that*

*(i)* $q_1^*(\bar{q}) \geq q^C \geq q_2^*(\bar{q})$, *with strict inequalities for* $\bar{q} > 2q^C$.

*(ii)* $q_1^*(\bar{q})$ *is strictly increasing in* $\bar{q}$ *and* $q_2^*(\bar{q})$ *is strictly decreasing in* $\bar{q}$.

*There is a cutoff* $\kappa \in (c'(0), p^C)$ *that is indirectly defined by*

$$\kappa \; : \; \kappa - c'\left(q_1^*(p^{-1}(\kappa))\right) \; = \; 0. \tag{12}$$

*It has the properties that*

*(iii)* $q_1^*(\bar{q})$ *and* $q_2^*(\bar{q})$ *are the unique equilibrium for all* $\bar{p} \in [\kappa, p^C]$.

*(iv)* $q_1^*(\bar{q})$ *and* $q_2^*(\bar{q})$ *are no equilibrium for all* $\bar{p} \in (c'(0), \kappa)$.

*Proof.* I begin by proving part (ii).

$q_1^*(\bar{q})$ *is strictly increasing in* $\bar{q}$ *and* $q_2^*(\bar{q})$ *is strictly decreasing in* $\bar{q}$.

The implicit function theorem yields the derivative of $q_2^*(\bar{q})$:

$$\frac{\partial q_2^*(\bar{q})}{\partial \bar{q}} \; = \; -\frac{p'(\bar{q}) + q_2^*(\bar{q}) \cdot p''(\bar{q})}{p'(\bar{q}) - c''(q_2^*(\bar{q}))}. \tag{A.6}$$

The denominator of this expression is negative by assumption. It remains to show that the numerator is negative, too.

Rearranging firm 2' first-order condition yields

$$q_2^*(\bar{q}) \; = \; \frac{p(\bar{q}) - c'(q_2^*(\bar{q}))}{p'(\bar{q})}. \tag{A.7}$$

Plugging this into the numerator of the derivative yields

$$p'(\bar{q}) + \frac{p(\bar{q}) - c'(q_2^*(\bar{q}))}{p'(\bar{q})} \cdot p''(\bar{q}) \; \overset{?}{<} \; 0 \tag{A.8}$$

$$\iff \quad (p(\bar{q}) - c'(q_2^*(\bar{q}))) \cdot p''(\bar{q}) - [p'(\bar{q})]^2 \; \overset{?}{<} \; 0, \tag{A.9}$$

where $\overset{?}{<}$ means that the inequality remains to be shown.

The strict log-concavity of $p(q)$ implies that

$$\forall q \; : \; p(q) \; > \; 0 \quad \Longrightarrow \quad p(q) \cdot p''(q) - [p'(q)]^2 \; < \; 0. \tag{A.10}$$

Since $p(\bar{q}) > 0$, this implies the above inequality because $c'(\cdot)$ is positive.

The derivative of $q_1^*(\bar{q})$ is

$$\frac{\partial q_1^*(\bar{q})}{\partial \bar{q}} \; = \; 1 - \frac{\partial q_2^*(\bar{q})}{\partial \bar{q}} \; > \; 0. \tag{A.11}$$

This proves part (ii).

I proceed by proving part (i).

$q_1^*(\bar{q}) \geq q^C \geq q_2^*(\bar{q})$, *with strict inequalities for* $\bar{p} < p^C$.

By definition, $\bar{q} \geq 2q^C$. In the corner case of $\bar{q} = 2q^C$, the equilibria in the game with a price cap and in the standard Cournot game coincide (because the price cap just binds): $q_1^*(2q^C) = q^C$ and $q_2^*(2q^C) = q^C$.

Because $q_1^*(\bar{q})$ is increasing, it follows that $q_1^*(\bar{q}) \geq q^C$.

Because $q_2^*(\bar{q})$ is decreasing, it follows that $q_2^*(\bar{q}) \leq q^C$.

This proves part (i).

Begin by assuming that $\kappa \in (c'(0), p^C)$ and verify at the end.

As $q_2^*(\bar{q})$ solves firm 2's first-order condition (and firm 2's profit function is strictly quasi-concave because of strategic substitutability), firm 2 has no profitable deviation. Whether $q_1^*(\bar{q})$ and $q_2^*(\bar{q})$ are an equilibrium, thus, depends only on whether firm 1 has a profitable deviation or not.

I proceed by proving part (iii).

I will show that $q_1^*(\bar{q})$ and $q_2^*(\bar{q})$ are an equilibrium if $\bar{p} \in [\kappa, p^C]$. Firm 1's marginal profit drops at $q_1^*(\bar{q})$ because the price cap stops binding, which introduces an inframarginal loss. Firm 1 has no profitable deviation if its marginal profit is weakly positive everywhere to the left of the drop at $q_1 + q_2 = \bar{q}$ and weakly negative everywhere to the right of the drop.

The marginal profit before the drop is given by the left derivative with respect to the own quantity, evaluated at $q_1 = q_1^*(\bar{q})$ and $q_2 = q_2^*(\bar{q})$,

$$\left.\frac{\partial_- \pi_1(q_1, q_2)}{\partial q_1}\right|_{q_1 = q_1^*(\bar{q}),\ q_2 = q_2^*(\bar{q})} = \bar{p} - c'(q_1^*(\bar{q})). \tag{A.12}$$

This expression is positive if the price cap is sufficiently large: In particular, if the price cap is the Cournot-Nash price, then $q_1^*(\bar{q}) = q_2^*(\bar{q}) = q^C$ and the left-derivative of the marginal profit is $p^C - c'(q^C) > 0$. This inequality follows from the Cournot-Nash equilibrium condition

$$p^C + \underbrace{q^C \cdot p'(2q^C)}_{<0} - c'(q^C) = 0.$$

If the price cap decreases, $q_1^*(\bar{q})$ increases continuously, as shown above. Thus, the marginal profit just above the drop decreases continuously. By definition of $\kappa$, the marginal profit is weakly positive for all $\bar{p} \in [\kappa, p^C]$.

The marginal profit after the drop is given by the right-derivative

$$\left.\frac{\partial_+ \pi_1(q_1, q_2)}{\partial q_1}\right|_{q_1 = q_1^*(\bar{q}),\ q_2 = q_2^*(\bar{q})} = p(\bar{q}) + q_1^*(\bar{q}) \cdot p'(\bar{q}) - c'(q_1^*(\bar{q})). \tag{A.13}$$

This is the same marginal profit as in the standard Cournot game without price caps. That this expression is weakly negative for all $\bar{p} \in (c'(0), p^C]$ can be shown by decomposing the change in the total quantity compared to the Cournot-Nash equilibrium. Specifically, keep the quantity of firm 2 at $q^C$ in the first step, while firm 1 brings the total quantity to $\bar{q}$. For this purpose, define $\hat{q_1}(\bar{q}) \equiv \bar{q} - q^C$. Because $\bar{q} \geq 2q^C$, it is true that $\hat{q_1}(\bar{q}) \geq q^C$. To show that the right-derivative of the marginal profit is negative, I use a set of inequalities (slightly abusing the notation, the marginal profits are those of the standard Cournot model; i.e. without a price cap):

$$0 = \frac{\partial \pi_1(q_1^C, q_2^C)}{\partial q_1} \geq \frac{\partial \pi_1\left(\hat{q_1}(\bar{q}), q_2^C\right)}{\partial q_1} \geq \frac{\partial \pi_1\left(q_1^*(\bar{q}), q_2^*(\bar{q})\right)}{\partial q_1}. \tag{A.14}$$

The equality follows from the Cournot-Nash equilibrium. The first inequality is true because the marginal profit is strictly quasi-concave in the own quantity. Moreover, if $\hat{q_1}(\bar{q}) > q^C$, the inequality is strict. To see that the second inequality is true, look at it written out (note that $\hat{q_1}(\bar{q}) + q^C = \bar{q} = q_1^*(\bar{q}) + q_2^*(\bar{q})$):

$$p(\bar{q}) + \hat{q_1}(\bar{q}) \cdot p'(\bar{q}) - c'(\hat{q_1}(\bar{q})) \geq p(\bar{q}) + q_1^*(\bar{q}) \cdot p'(\bar{q}) - c'(q_1^*(\bar{q})). \tag{A.15}$$

The inequality is true because $p'(\bar{q})$ is negative and because $q_1^*(\bar{q}) \geq \hat{q}_1(\bar{q})$. Moreover, if $q_1^*(\bar{q}) > \hat{q}_1(\bar{q})$, the inequality is strict. This implies that the marginal profit below the drop is weakly negative for all price caps.

Therefore, $q_1^*(\bar{q})$ and $q_2^*(\bar{q})$ are an equilibrium if $\bar{p} \in [\kappa, p^C]$.

Furthermore, this equilibrium is unique in this range:

For $q_1 + q_2 = \bar{q}$, there can be no other equilibrium because the solution to the first-order condition of firm 2 is unique.

For $q_1 + q_2 > \bar{q}$, Proposition 1 shows that there is no equilibrium.

For $q_1 + q_2 < \bar{q}$, there exists no equilibrium in this range of price caps: The first-order condition of firm 2 implies that, due to strategic substitutability, the optimal $q_2 > q_2^*(\bar{q})$. Therefore, to not violate $q_1 + q_2 < \bar{q}$, it has to be that $q_1 < q_1^*(\bar{q})$. This, however, implies that the marginal profit of firm 1 is strictly positive as $\bar{p} - c'(q_1) > \bar{p} - c'(q_1^*(\bar{q})) \geq 0$. The first inequality is true because $c'(\cdot)$ is strictly increasing. The second inequality is shown to be true above. Therefore, firm 1 could profitably deviate by expanding its quantity.

This proves part (iii).

I proceed by proving part (iv).

I will show that $q_1^*(\bar{q})$ and $q_2^*(\bar{q})$ are no equilibrium if $\bar{p} \in (c'(0), \kappa)$. As shown above, the left-derivative of firm 1's marginal profit at $q_1^*(\bar{q})$ and $q_2^*(\bar{q})$ is strictly decreasing in $\bar{q}$; so it is strictly increasing in $\bar{p}$. At $\bar{p} = \kappa$, the left-derivative of the marginal profit is 0. If $\bar{p} < \kappa$, it is negative. Therefore, firm 1 could profitably deviate by reducing its quantity.

This proves part (iv).

I conclude by proving that $\kappa \in (c'(0), p^C)$.

By definition, $\kappa$ is the price cap for which the left-derivative of firm 1's marginal profit evaluated at $q_1^*(\bar{q})$ and $q_2^*(\bar{q})$ is 0. As shown above, the marginal profit given these quantities is strictly increasing in $\bar{p}$ and it is strictly positive at $p^C$. Thus, $\kappa < p^C$.

To show that $\kappa > c'(0)$, I show that when the price cap is $c'(0)$, the suggested quantities are no equilibrium because firm 1 could profitably deviate by reducing its quantity: Its marginal profit above the drop given the suggested equilibrium quantities is strictly negative. The idea is that the equilibrium price is the marginal cost of the first unit, but firm 1's suggested equilibrium quantity is positive. Plugging the price cap into the suggested equilibrium quantities yields $q_2^*(p^{-1}(c'(0))) = 0$ and, hence, $q_1^*(p^{-1}(c'(0))) = p^{-1}(c'(0)) > 0$. Thus, firm 1's marginal profit above the drop is $c'(0) - c'(p^{-1}(c'(0)))$, which is negative because $c'(\cdot)$ is strictly increasing.

□

### A.3 Proof of Proposition 2

*Define the competitive price, $p^W$, as the price at which the inverse demand curve, $p(q)$, and the social marginal cost curve, $c'(\frac{q}{2})$, intersect.*

*Define $p^S$ as the unique Stackelberg equilibrium price if the Stackelberg leader's profit function is strictly* *quasi-concave.*

*It is true that* $p^W < \kappa < p^S$.

*Proof.* I first prove that $p^W < \kappa$ by contradiction.

Define $p^{-1}(p^W) \equiv q^W$.

Assume that there is a clearing equilibrium with $\bar{p} = p^W$. One of the optimality conditions of firm 1 says that the marginal profit just above the drop has to be weakly positive,

$$p^W - c'(q_1^*) \;\geq\; 0. \tag{A.16}$$

This expression is strictly decreasing in $q_1^*$. If $q_1^* = \frac{q^W}{2}$, then the inequality has to bind because of the definition of the competitive price, $p^W$. Thus, firm 1 can never produce more than $\frac{q^W}{2}$ in any clearing equilibrium.

Because the in the clearing equilibrium the total quantity has to be $q^W$, firm 2 has to produce at least $\frac{q^W}{2}$. This, however, cannot be optimal for firm 2, as $\frac{q^W}{2} > q^C$, but Theorem 1 has shown that firm 2's suggested equilibrium quantity for such a total quantity has to be less than $q^C$. Firm 2's optimality condition fails because of the inframarginal loss,

$$p^W - c'(q_2^*) + q_2^* \cdot p'(q^W) \;=\; 0. \tag{A.17}$$

The first two terms together are at most 0 (if $q_2^* = \frac{q^W}{2}$). The third term, the inframarginal loss, is strictly negative. Thus, there can be no clearing equilibrium if the price cap is the competitive price.

I now prove that $\kappa < p^S$.

The Stackelberg equilibrium $(s_1^*, s_2(s_1^*))$ is defined by

$$p(s_1^* + s_2(s_1^*)) - c'(s_1^*) + \underbrace{s_1^*}_{>0} \cdot \underbrace{p'(s_1^* + s_2(s_1^*))}_{<0} \cdot \underbrace{\left(1 + \frac{\partial s_2(s_1^*)}{\partial s_1}\right)}_{>0} \stackrel{!}{=} 0. \tag{A.18}$$

The cutoff $\kappa$ is defined by

$$p(q_1^*(p^{-1}(\kappa)) + q_2^*(p^{-1}(\kappa))) - c'(q_1^*(p^{-1}(\kappa))) \;=\; 0. \tag{A.19}$$

Combining these equations yields

$$p(s_1^* + s_2(s_1^*)) - c'(s_1^*) \;>\; p(q_1^*(p^{-1}(\kappa)) + q_2^*(p^{-1}(\kappa))) - c'(q_1^*(p^{-1}(\kappa))). \tag{A.20}$$

Now use two facts:

(i) On both sides are values of the same function of $q_1$: Because the Stackelberg follower and firm 2 both have the best response function of a standard Cournot duopolist as their best response function, they respond with the same quantities to the same $q_1$.

(ii) This function of $q_1$ is strictly decreasing in $q_1$: The price falls in $q_1$ because the total quantity increases in $q_1$ (the best response function's slope is larger than $-1$, see Proposition 1) and the marginal cost is increasing in $q_1$.

Therefore, the inequality implies that $s_1^* < q_1^*(p^{-1}(\kappa))$, so the total quantity in the unique Stackelberg equilibrium is strictly less than in the clearing equilibrium at the cutoff. So, the Stackelberg equilibrium price is strictly larger than the cutoff, $p^S > \kappa$.

□

### A.4 Proof of Theorem 2

*If $\bar{p} \in (c'(0), \kappa)$, the only pure-strategy Nash equilibrium is*

$$q_1^*(\bar{p}) \;=\; (c')^{-1}(\bar{p}) \quad \textit{and} \tag{17}$$

$$q_2^*(\bar{p}) \;:\; p\Big(q_1^*(\bar{p}) + q_2^*(\bar{p})\Big) + q_2^*(\bar{p}) \cdot p'\Big(q_1^*(\bar{p}) + q_2^*(\bar{p})\Big) - c'(q_2^*(\bar{p})) \;\stackrel{!}{=}\; 0. \tag{18}$$

*In the limit of $\bar{p} \to \kappa$, the equilibrium converges to the clearing equilibrium presented in Theorem 1.*

*$q_1^*(\bar{p})$ is strictly increasing in $\bar{p}$ and $q_2^*(\bar{p})$ is strictly decreasing in $\bar{p}$. The total quantity $q_1^*(\bar{p}) + q_2^*(\bar{p})$ is strictly increasing in $\bar{p}$.*

*Proof.* As $\bar{p} < \kappa$, a clearing equilibrium with $q_1 + q_2 = \bar{q}$ does not exist (see Theorem 1). Proposition 1 shows that no equilibrium exists with $q_1 + q_2 > \bar{q}$. So, if an equilibrium exists, it has to be that $q_1 + q_2 < \bar{q}$.

Firm 1's optimality condition then prescribes that its marginal profit has to get negative before the drop. Its profit-maximization condition is $\bar{p} = c'(q_1)$. So, the unique candidate for an equilibrium strategy is $q_1 = (c')^{-1}(\bar{p})$. Note that this candidate strategy converges to the equilibrium strategy in the clearing equilibria at the cutoff, $\lim_{\bar{p}\to\kappa}(c')^{-1}(\bar{p}) = q_1^*(p^{-1}(\kappa))$, as both solve $\kappa = c'(q_1)$.

Plugging firm 1's optimal choice into the first-order condition of firm 2 yields

$$p(q_1^*(\bar{p}) + q_2) + q_2 \cdot p'(q_1^*(\bar{p}) + q_2) - c'(q_2) \;\stackrel{!}{=}\; 0. \tag{A.21}$$

This equation has a solution because the left-hand side is continuous and it is positive at $q_2 = 0$ (because $p(q_1^*(\bar{p})) > c'(0)$) and negative for large $q_2$, for example at $q_2 = p^{-1}(c'(0))$. The solution is unique because strategic substitutability makes firm 2's profit function strictly quasi-concave.

The comparative statics of firm 1's quantity follow from the inverse function theorem:

$$\frac{\partial q_1^*(\bar{p})}{\partial \bar{p}} \;=\; \frac{\partial (c')^{-1}(\bar{p})}{\partial \bar{p}} \;=\; \frac{1}{c''\left((c')^{-1}(\bar{p})\right)} \;>\; 0. \tag{A.22}$$

The comparative statics of firm 2's quantity follow from a decomposition: The choice of firm 2 only depends on the price cap via the choice of firm 1. Thus, the derivative is given by the product of the slope of firm 2's best response function (which is strictly between −1 and 0) and the change in firm 1's choice,

$$\frac{\partial q_2^*(\bar{p})}{\partial \bar{p}} \;=\; \underbrace{\frac{\partial q_2^*(\bar{p})}{\partial q_1}}_{<0} \cdot \underbrace{\frac{\partial q_1^*(\bar{p})}{\partial \bar{p}}}_{>0} \;<\; 0. \tag{A.23}$$

The same decomposition can be used to determine the comparative statics of the total equilibrium quantity,

$$\frac{\partial (q_1^*(\bar{p}) + q_2^*(\bar{p}))}{\partial \bar{p}} \;=\; \underbrace{\frac{\partial q_1^*(\bar{p})}{\partial \bar{p}}}_{>0} \cdot \underbrace{\left(1 + \frac{\partial q_2^*(\bar{p})}{\partial q_1}\right)}_{>0} \;>\; 0. \tag{A.24}$$

□

### A.5 Proof of Lemma 1

*Define $\bar{p}_B$ as the price cap for which the total quantity in the rationing equilibrium is equal to the Cournot-Nash quantity, $\bar{p}_B : q_1^*(\bar{p}_B) + q_2^*(\bar{p}_B) = 2q^C$. It is true that $c'(0) < \bar{p}_B < \kappa$.*

*There is a monotone bijection between the clearing equilibria and the rationing equilibria. For each price cap $\bar{p}_c \in (\kappa, p^C]$, there is exactly one price cap $\bar{p}_r \in [\bar{p}_B, \kappa)$ such that the equilibrium quantities of the firms are the same:*

$$q_1^*(\bar{q}_c) = q_1^*(\bar{p}_r) \qquad \textit{and} \qquad q_2^*(\bar{q}_c) = q_2^*(\bar{p}_r). \tag{19}$$

*Proof.* Because the slope of firm 2's best response function lies strictly between 0 and -1, for any fixed total quantity, $q_1 + q_2$, there is at most one possible split between $q_1$ and $q_2$ such that $q_2$ is the best response to $q_1$. Because firm 2's profit-maximizing problem is the same, this fact is true for both clearing and rationing equilibria. So, if the total quantity in a clearing equilibrium and in a rationing equilibrium are equal, it has to be that also the firms' individual quantities are equal.

The price cap $\bar{p}_B$ exists and is unique because the total quantity in equilibrium in the rationing equilibria is continuous and monotone. When the price cap goes to $c'(0)$, the total quantity goes to the monopoly quantity $q^M$. When the price cap goes to $\kappa$, the total quantity exceeds $2q^C$, as the clearing and the rationing equilibrium converge at $\kappa$ (see Theorem 2). Because in the clearing equilibria the total quantity is $\bar{q} = p^{-1}(\bar{p})$ and $\bar{p} = \kappa < p^C$, the total quantity exceeds $2q^C$.

The range of total quantities in the equilibria is the same for price caps in $(\kappa, p^C]$ and for price caps in $[\bar{p}_B, \kappa)$ because the total quantities are the same for the extreme points and in-between the total quantity is monotone and continuous. In the extreme points, the total quantities are the same by construction of the intervals: Theorem 2 has shown that the equilibrium quantities converge for the price cap's going to $\kappa$ from above in clearing and from below in rationing equilibria. The price cap $\bar{p}_B$ is defined such that the total quantity in the rationing equilibrium is equal to the total quantity in the clearing equilibrium with price cap $p^C$. The monotonicity and continuity of the equilibrium quantities follows in the clearing equilibria from the fact that the total quantity is $\bar{q}$, which is monotone and continuous in $\bar{p}$, and in the rationing equilibria, it has been shown in Theorem 2.

The monotonicity also implies that the bijection between $\bar{p}_c$ and $\bar{p}_r$ is monotone. Thus, if there is a pair of price caps that induce the same equilibrium quantities $\bar{p}_c^1$ and $\bar{p}_r^1$ and another such pair $\bar{p}_c^2$ and $\bar{p}_r^2$ with $\bar{p}_c^1 > \bar{p}_c^2$, then it has to be that $\bar{p}_r^1 < \bar{p}_r^2$.

□

### A.6 Proof of Proposition 3

*When the marginal cost is constant at c, the unique equilibrium is*

$$q_1^*(\bar{q}) = \bar{q} - q_2^*(\bar{q}) \quad \textit{and} \quad q_2^*(\bar{q}) = \frac{p(\bar{q}) - c}{-p'(\bar{q})}. \tag{25}$$

*Proof.* First, note that with constant marginal cost, the range of price caps becomes $(c, p^C]$, where, re-purposing notation, $p^C$ is the Cournot-Nash price with constant marginal cost.

With constant marginal cost, firm 1's marginal profit becomes (expressed as the right-derivative at the drop $q_1+q_2=\bar{q}$)

$$\frac{\partial_+\pi_1(q_1,q_2)}{\partial q_1} = \begin{cases} \bar{p}-c & \text{if } q_1+q_2 < \bar{q} \\ p(q_1+q_2)+q_1\cdot p'(q_1+q_2)-c & \text{if } q_1+q_2 \geq \bar{q}. \end{cases} \tag{A.25}$$

The first step is to prove that there can be no equilibrium that is not clearing.

If $q_1+q_2<\bar{q}$, firm 1's marginal profit is strictly positive, so it could profitably deviate by increasing $q_1$.

If $q_1+q_2>\bar{q}$, Proposition 1 (at least one firm has a strictly negative marginal profit if $q_1+q_2>\bar{q}$) can be adjusted to the case of constant marginal cost: Simply drop $c''(q_i^*)$ from Equation (A.5).

Hence, in all equilibria it has to be that $q_1+q_2=\bar{q}$, which means, that they are clearing equilibria. The equilibrium strategies follow analogously to Theorem 1:

Plugging the equilibrium condition $q_1+q_2=\bar{q}$ into firm 2's first-order condition yields

$$p(\bar{q})+q_2\cdot p'(\bar{q})-c = 0 \iff q_2^*(\bar{q}) = \frac{p(\bar{q})-c}{-p'(\bar{q})}, \tag{A.26}$$

which has a unique solution because firm 2's profit function is strictly quasi-concave.

Rearranging the equilibrium condition $q_1+q_2=\bar{q}$ yields a unique candidate for the equilibrium quantity of firm 1,

$$q_1^*(\bar{q}) = \bar{q}-q_2^*(\bar{q}). \tag{A.27}$$

Showing that this candidate is actually an equilibrium is, again, analogous to Theorem 1: Firm 1's marginal profit is weakly positive above and weakly negative below the drop. The marginal profit above the drop is $\bar{p}-c>0$. That the marginal profit below the drop is weakly negative follows from the same decomposition as in Theorem 1.

By the same argument as in Theorem 1, it is true that $\frac{\partial q_2^*(\bar{q})}{\partial \bar{q}}<0$, so $q_2^*(\bar{q})\leq q^C$.

Define $\hat{q_1}(\bar{q})\equiv\bar{q}-q^C$, for which it is true that $\hat{q_1}(\bar{q})\geq q^C$. Then,

$$0 = \frac{\partial\pi_1(q_1^C,q_2^C)}{\partial q_1} \geq \frac{\partial\pi_1\left(\hat{q_1}(\bar{q}),q_2^C\right)}{\partial q_1} \geq \frac{\partial\pi_1\left(q_1^*(\bar{q}),q_2^*(\bar{q})\right)}{\partial q_1}. \tag{A.28}$$

Again, slightly abusing the notation, the marginal profits denote the marginal profits of standard Cournot duopolists. The equality follows from the Cournot-Nash equilibrium. The first inequality is true because the marginal profit is strictly quasi-concave in the own quantity. To see that the second inequality is true, look at it written out (note that $\hat{q_1}(\bar{q})+q^C=\bar{q}=q_1^*(\bar{q})+q_2^*(\bar{q})$):

$$p(\bar{q})+\hat{q_1}p'(\bar{q})-c \geq p(\bar{q})+q_1^*p'(\bar{q})-c. \tag{A.29}$$

The inequality is true because $p'(\bar{q})$ is negative and because $q_1^*(\bar{q})\geq\hat{q_1}(\bar{q})$. Moreover, if $q_1^*(\bar{q})>\hat{q_1}(\bar{q})$, the inequality is strict.

□

### A.7 Proof of Proposition 4

*Assume that firm 1 has the price cap $\bar{p}$ and firm 2 has a price cap $\bar{p}_2$. Without loss of generality, assume that $\bar{p} < \bar{p}_2$.*

*Define $\kappa_2(\bar{p})$ as the value at which $p(q_1^*(\bar{p}) + q_2)$ and $c'(q_2)$—both functions of $q_2$—intersect; illustrated in Figure 14. $\kappa_2(\bar{p})$ is decreasing in $\bar{p}$.*

*(i) If $\bar{p} \geq \kappa$, the only pure-strategy Nash equilibrium is the clearing equilibrium described in Theorem 1. It does not depend on $\bar{p}_2$.*

*(ii) If $\bar{p} < \kappa$ and $\bar{p}_2 \geq p(q_1^*(\bar{p}) + q_2^*(\bar{p}))$ (as defined in Theorem 2), the only pure-strategy Nash equilibrium is the rationing equilibrium described in Theorem 2. It does not depend on $\bar{p}_2$.*

*(iii) If $\bar{p} < \kappa$ and $p(q_1^*(\bar{p}) + q_2^*(\bar{p})) > \bar{p}_2 \geq \kappa_2(\bar{p})$, the only pure-strategy Nash equilibrium is a partially rationing equilibrium. In this equilibrium, firm 1 produces $q_1^*(\bar{p}) = (c')^{-1}(\bar{p})$ and firm 2 produces the quantity that brings the market-clearing price to $\bar{p}_2$, which is $q_2^*(\bar{p}, \bar{p}_2) = p^{-1}(\bar{p}_2) - q_1^*(\bar{p})$.*

*(iv) If $\bar{p} < \kappa$ and $\kappa_2(\bar{p}) > \bar{p}_2$, the only pure-strategy Nash equilibrium is a doubly rationing equilibrium. In this equilibrium, firm 1 produces $q_1^*(\bar{p}) = (c')^{-1}(\bar{p})$ and firm 2 produces $q_2^*(\bar{p}_2) = (c')^{-1}(\bar{p}_2)$, which depends only on the own price cap.*

*Proof.* Define $\bar{q}_2 \equiv p^{-1}(\bar{p}_2)$.

The price cap of firm 2 has the same effect as for firm 1 as long as it binds, so the marginal profit becomes

$$\frac{\partial_+ \pi_2(q_1, q_2)}{\partial q_2} = \begin{cases} \bar{p}_2 - c'(q_2) & \text{if } q_1 + q_2 < \bar{q}_2 \\ p(q_1 + q_2) + q_2 \cdot p'(q_1 + q_2) - c'(q_2) & \text{if } q_1 + q_2 \geq \bar{q}_2. \end{cases} \tag{A.30}$$

Case (i): At the equilibrium quantities, the price cap of firm 2 does not bind because the price is $\bar{p}$. Firm 1 does not want to deviate because nothing has changed in comparison to Theorem 1. Firm 2 does not want to deviate because for all smaller quantities, the marginal profit is strictly positive, and for all larger quantities, the marginal profit is strictly negative.

The uniqueness follows from Proposition 1 (no equilibrium with $q_1 + q_2 > \bar{q}$) and Theorem 1 (no other equilibrium with $\bar{q}_2 < q_1 + q_2 \leq \bar{q}$) for all total quantities, for which the price cap of firm 2 does not bind.

For all quantities for which the price cap of firm 2 does bind, there is a decomposition argument similar to the one used in Theorem 1 that shows that firm 2 always has a profitable deviation. To derive a contradiction, assume that $q_1'$ and $q_2'$ constitute an equilibrium, such that $q_1' + q_2' \leq \bar{q}_2$.

Firm 1's optimality condition, because its price cap strictly binds, is that

$$\bar{p} - c'(q_1') = 0. \tag{A.31}$$

In the equilibrium with $q_1^* + q_2^* = \bar{q}$, firm 1's marginal profit above the drop is weakly positive,

$$\bar{p} - c'(q_1^*(\bar{q})) \geq 0. \tag{A.32}$$

As the marginal cost is strictly increasing, both conditions combined imply that $q_1' \geq q_1^*(\bar{q})$.

Since the total quantity has to be smaller in the fictitious equilibrium, it has to hold that $q_2' < q_2^*(\bar{q})$.

Define $\hat{q_2}(\bar{q}) \equiv \bar{q}_2 - q_1^*(\bar{q})$. It is true that $q_2' \leq \hat{q_2}(\bar{q}) < q_2^*(\bar{q})$. The first inequality follows from plugging in for $\bar{q}_2 \geq q_1' + q_2'$ in the definition of $\hat{q_2}(\bar{q})$ and then applying the inequality $q_1' \geq q_1^*(\bar{q})$. The second inequality follows from $\hat{q_2}(\bar{q}) = \bar{q}_2 - q_1^*(\bar{q}) < \bar{q} - q_1^*(\bar{q}) = q_2^*(\bar{q})$.

Applying this decomposition of the difference between $q_2'$ and $q_2^*(\bar{q})$ to the marginal profit of a standard Cournot duopolist, that is, including the inframarginal loss, yields

$$0 \;=\; \frac{\partial \pi_2\big(q_1^*(\bar{q}), q_2^*(\bar{q})\big)}{\partial q_2} \;<\; \frac{\partial \pi_2\big(q_1^*(\bar{q}), \hat{q_2}(\bar{q})\big)}{\partial q_2} \;\leq\; \frac{\partial \pi_2\big(q_1', q_2'\big)}{\partial q_2}. \tag{A.33}$$

The equality follows from the optimality condition in the confirmed equilibrium. The first inequality follows from strict quasi-concavity of the profit function because the quantity of firm 2 is strictly reduced. In the step to the right-hand side, the total quantity is kept fixed, while the quantity of firm 1 is weakly increased and the quantity of firm 2 weakly decreased. The marginal profit for a fixed total quantity is decreasing in the own quantity because $p'(\bar{q}_2)$ is strictly negative and $c'(\cdot)$ is strictly increasing. So if $q_1' > q_1^*(\bar{q})$, the inequality is strict.

The fictitious equilibrium, however, is only an equilibrium if the right-hand side is weakly negative: It is firm 2's marginal profit below the drop. As it is strictly positive, firm 2 profits from deviating to a larger quantity. Thus, $(q_1', q_2')$ are no equilibrium.

The other three cases correspond to rationing equilibria in the case when only firm 1 has a price cap. In the rationing equilibria, the game is essentially nonstrategic and firm 2 is a monopolist on the market for residual demand. Thus, the three cases directly correspond to the three cases in monopoly regulation (see Appendix B). The equilibrium price when firm 2 has no price cap, $p(q_1^*(\bar{p}) + q_2^*(\bar{p}))$, corresponds to the monopoly price, $p^M$. A cutoff that depends on the quantity of firm 1, $\kappa_2(\bar{p})$, corresponds to the competitive price.

For the following three cases, the uniqueness argument is the same: Proposition 1 and Theorem 1 show that there can be no pure-strategy Nash equilibria in which $q_1 + q_2 \geq \bar{q}$, so in any equilibrium, it has to be that firm 1's price cap strictly binds (that is, firm 1 is not at the drop of its marginal revenue). Because firm 1's price cap strictly binds in any equilibrium, the only optimal strategy of firm 1 is to play $q_1^*(\bar{p}) = (c'^{-1})(\bar{p})$. Given that firm 1 plays $q_1^*(\bar{q})$ in any equilibrium, the uniqueness of the equilibria follows from the uniqueness of firm 2's best response.

Case (ii): This corresponds to the case in the monopoly in which the price cap is so high that it makes no difference. The monopolist's marginal profit intersects zero at a quantity beyond the drop.

The equilibrium price is $p(q_1^*(\bar{p}) + q_2^*(\bar{p})) \leq \bar{p}_2$, so the additional constraint of firm 2's price cap is not violated in the solution, in which firm 2 ignores its price cap. Thus, this solution remains optimal and the price cap $\bar{p}_2$ makes no difference.

For the following two cases, the competitive price on the market for the residual demand, $\kappa_2(\bar{p})$, is needed. Given that firm 1 produces the quantity $q_1^*(\bar{p}) = (c')^{-1}(\bar{p})$, firm 2's inverse residual demand curve and its marginal cost curve intersect at the price $\kappa_2(\bar{p})$. Figure 14 illustrates the principle. Formally,

$$\kappa_2(\bar{p}) \;\equiv\; p(q_1^*(\bar{p}) + q_2^*(\bar{p})), \tag{A.34}$$

$$\text{where } q_2^*(\bar{p}) \text{ solves } \quad p(q_1^*(\bar{p}) + q_2^*(\bar{p})) - c'(q_2^*(\bar{p})) \stackrel{!}{=} 0. \tag{A.35}$$

The comparative statics follow from the comparative statics of firm 1's equilibrium quantity. If $\bar{p}$ increases, $q_1^*(\bar{p})$ increases, so firm 2's inverse residual demand function is shifted to the left (decreases). Therefore, the inverse residual demand curve and the marginal cost curve intersect at a lower price, $\kappa_2(\bar{p})$.[24]

Case (iii): This corresponds to the case in the monopoly in which the price cap is between the monopoly price and the competitive price, so it binds, and the price clears the market. The monopolist's marginal profit intersects zero at the drop.

The total quantity for the proposed strategies is $\bar{q}_2$, that is, firm 2's price cap just binds, and firm 2 is at the drop in its marginal profit. Firm 1 does not want to deviate because nothing has changed in comparison to Theorem 2. Firm 2 has no incentive to deviate either.

Its marginal profit above the drop is weakly positive. By definition of $\kappa_2(\bar{p})$, given $q_1^*(\bar{p})$, the quantity $q_2'$ that equates the inverse demand and the competitive price on the market for residual demand, $p(q_1^*(\bar{p})+q_2')=\kappa_2(\bar{p})$ solves also $p(q_1^*(\bar{p})+q_2')-c'(q_2')=0$. This latter function is strictly decreasing in $q_2$. By the case condition, $p(q_1^*(\bar{p})+q_2^*(\bar{p},\bar{p}_2))=\bar{p}_2\geq\kappa_2(\bar{p})$, hence $q_2^*(\bar{p},\bar{p}_2)\leq q_2'$. The marginal profit above the drop is $\bar{p}_2-c'(q_2^*(\bar{p},\bar{p}_2))\geq\kappa_2(\bar{p})-c'(q_2')\equiv 0$ because $\bar{p}_2\geq\kappa_2(\bar{p})$ and $c'(q_2^*(\bar{p},\bar{p}_2))\leq c'(q_2')$.

After the drop and for all larger quantities, firm 2's marginal profit is strictly negative. This follows from three previously established and one new fact. First, after the drop, the price cap does not bind, so firm 2's marginal profit is the same as that of a standard Cournot duopolist. Second, the profit function of a standard Cournot duopolist is strictly quasi-concave in the own quantity. Thirdly, given $q_1^*(\bar{p})$, the root of the standard Cournot duopolist's marginal profit is given by $q_2^*(\bar{p})$, as shown in Theorem 2. Lastly, $q_2^*(\bar{p},\bar{p}_2)>q_2^*(\bar{p})$ because $p(q_1^*(\bar{p})+q_2^*(\bar{p},\bar{p}_2))=\bar{p}_2<p(q_1^*(\bar{p})+q_2^*(\bar{p}))$, where the inequality follows from the case condition.

Case (iv): This corresponds to the case in the monopoly in which the price cap is between the competitive price and the marginal cost of the first unit, so the price cap binds, and the price does not clear the market. The monopolist's marginal profit intersects zero at a quantity before the drop.

At the proposed quantities, both price caps bind. Firm 1 does not want to deviate because nothing has changed in comparison to Theorem 2. Firm 2 has no incentive to deviate because its marginal profit intersects zero already before the drop. As shown above, the marginal profit above the drop is $\bar{p}_2-c'(q_2^*(\bar{p},\bar{p}_2))<\kappa_2(\bar{p})-c'(q_2')\equiv 0$, where $q_2'$ follows from the definition of the competitive price on the market for residual demand, $p(q_1^*(\bar{p})+q_2')=\kappa_2(\bar{p})$. The inequality follows from the case condition $\bar{p}_2<\kappa_2(\bar{p})$, which also implies that $q_2^*(\bar{p},\bar{p}_2)>q_2'$. Thus, firm 2's marginal profit intersects zero at $q_2^*(\bar{p}_2)=(c')^{-1}(\bar{p}_2)$ and the marginal profit is strictly positive for all smaller quantities and strictly negative for all larger quantities.

□

### A.8 Proof of Proposition 6

*(i) There are no equilibria with $q_1+q_2>\bar{q}$ (Proposition 1 remains true with proportional rationing).*

24. In the monopoly analogy, the competitive price gets lower when the inverse demand function is shifted to the left.

*(ii) There are no clearing equilibria with $q_1 + q_2 = \bar{q}$.*

*(iii) There are rationing equilibria with $q_1 + q_2 < \bar{q}$. There is an $\epsilon > 0$, such that the unique equilibrium for each price cap in $(c'(0), \kappa + \epsilon)$ is a rationing equilibrium. When the price cap decreases, firm 1 decreases and firm 2 increases its quantity.*

*Proof.* Firm 1's best response function is the same as in the case of efficient rationing because it does not depend on the rationing rule (the quantity of firm 2 is efficiently rationed because the price is free to adjust).

(i) When $q_1 + q_2 > \bar{q}$, both firms are standard Cournot duopolists as the price cap does not bind. Thus, Proposition 1 remains true.

(ii) Because firm 2's marginal profit jumps at $q_2 = \bar{q} - q_1$, this quantity is never optimal for firm 2.

There are four sub-cases. Remember that firm 2's profit function is strictly quasi-concave in $q_2$ within both parts and that the marginal profit crosses zero at least once.

(ii.i) If the marginal profit jumps from the strictly negative into the weakly negative, a strictly lower quantity is optimal.

(ii.ii) If the marginal profit jumps from the weakly positive into the strictly positive, a strictly larger quantity is optimal.

(ii.iii) If the marginal profit jumps from the weakly negative into the strictly positive, it is both better to choose a slightly smaller or larger quantity.

(ii.iv) If the marginal profit has no jump, which means that $q_1 = 0$, the monopoly quantity, which is strictly less than $\bar{q}$, is optimal.

(iii) As before, in any rationing equilibrium, firm 1 plays $q_1^*(\bar{p}) = (c')^{-1}(\bar{p})$.

There is a rationing equilibrium if the best response functions intersect in the vertical part of firm 1's best response function. That is, firm 2's best response to $q_1^*(\bar{p})$ is such that $q_2 \leq \bar{q} - q_1^*(\bar{p})$. The global maximizer of firm 2's two-part profit function is in the first part. I now show that a rationing equilibrium exists for all price caps $\bar{p} < \kappa$.

The second part of firm 2's profit function is the profit function of a standard Cournot duopolist. If $\bar{p} < \kappa$, the best response of a standard Cournot duopolist is $BR_2(q_1^*(\bar{p})) < \bar{q} - q_1^*(\bar{p})$. This inequality is necessarily true because, with efficient rationing, a rationing equilibrium exists for these price caps (see Theorem 2). Furthermore, as the standard Cournot duopolist's profit function is strictly quasi-concave in $q_2$, its marginal profit is strictly negative for all $q_2 > BR_2(q_1^*(\bar{p}))$. So, in particular, the marginal profit is strictly negative for $q_2 = \bar{q} - q_1^*(\bar{p})$. Or, expressed in terms of firm 2 with proportional rationing and its two-part profit function: The right-derivative of firm 2's profit with respect to $q_2$ is strictly negative in the second part at the transition point at $\bar{q} - q_1^*(\bar{p})$.

The above fact and the fact that the profit function of firm 2 is strictly quasi-concave in $q_2$ within each of the two parts, implies that the global maximizer of firm 2's profit is in the first part of its two-part profit function: As firm 2's marginal profit is strictly negative above the jump at $q_2 = \bar{q} - q_1^*(\bar{p})$, it is also strictly negative below the jump. Thus, the only intersection of firm 2's marginal profit and zero is in the first part of the profit function.

Therefore, $q_1^*(\bar{p})$ and the $q_2$ that solves

$$p\left(\frac{\bar{q}}{\bar{q}-q_1^*(\bar{p})}\cdot q_2\right)+q_2\cdot\frac{\bar{q}}{\bar{q}-q_1^*(\bar{p})}\cdot p'\left(\frac{\bar{q}}{\bar{q}-q_1^*(\bar{p})}\cdot q_2\right)-c'(q_2)\overset{!}{=}0 \tag{A.36}$$

constitute a rationing equilibrium.

Moreover, continuity implies that there are also rationing equilibria for some price caps above the cutoff $\kappa$.

For the price cap $\kappa$, there is a rationing equilibrium: By definition of $\kappa$, it is true that $BR_2(q_1^*(\kappa)) = p^{-1}(\kappa)-q_1^*(\kappa)$. So, firm 2's marginal profit is 0 above the jump at the transition point at $p^{-1}(\kappa)-q_1^*(\kappa)$. Below the jump, the marginal profit is strictly negative. Thus, the maximum in the first part of the profit function is strictly larger than the corner maximum in the second part of the profit function.

When marginally increasing the price cap, the global maximum remains in the first part of firm 2's profit function because $q_1(\bar{p})$ changes continuously and the two candidates for firm 2's best response change continuously, so the corresponding profits change continuously. As the profit in the maximum of the first part was strictly larger than the profit in the maximum of the second part, it remains strictly larger for some larger price caps. Then, $q_1^*(\bar{p})$ and the $q_2$ described by (A.36) still are a rationing equilibrium.

The comparative statics of the equilibrium quantity of firm 1, $q_1^*(\bar{p})$, with respect to the price cap are the same as with efficient rationing. The comparative statics of $q_2^*(\bar{p})$ follow from applying the implicit function theorem on firm 2's first-order condition (A.36). I take the derivative with respect to $\bar{q}=p^{-1}(\bar{p})$. The derivative with respect to $\bar{p}$ has the opposite sign. The derivative is, slightly abusing the $\bar{p}$ and $\bar{q}$ notation,

$$\frac{\partial q_2^*(\bar{p})}{\partial \bar{q}} = -q_2^*(\bar{p})\cdot\alpha\cdot\frac{\gamma}{\beta\cdot\gamma-c''(q_2^*)}, \tag{A.37}$$

where I substituted to improve readability:

$$\alpha \equiv \frac{\bar{q}\cdot\frac{\partial q_1^*(\bar{p})}{\partial\bar{q}}-q_1^*(\bar{p})}{(\bar{q}-q_1)^2} < 0, \tag{A.38}$$

$$\beta \equiv \frac{\bar{q}}{\bar{q}-q_1^*(\bar{p})} > 1, \quad \text{and} \tag{A.39}$$

$$\gamma \equiv 2p'(\beta\cdot q_2^*)+q_2^*\cdot\beta\cdot p''(\beta\cdot q_2^*) < 0. \tag{A.40}$$

The bound for $\alpha$ follows from $\frac{\partial q_1^*(\bar{p})}{\partial\bar{q}}<0$. The bound for $\gamma$ follows from the strict log-concavity of firm 2's inverse residual demand function in $q_2$. Thus, whenever $q_2^*(\bar{p})>0$, then

$$\frac{\partial q_2^*(\bar{p})}{\partial\bar{q}} > 0. \tag{A.41}$$

□

## Appendix B Price Regulation in Monopoly

This section summarizes the textbook case of price regulation in a monopoly.

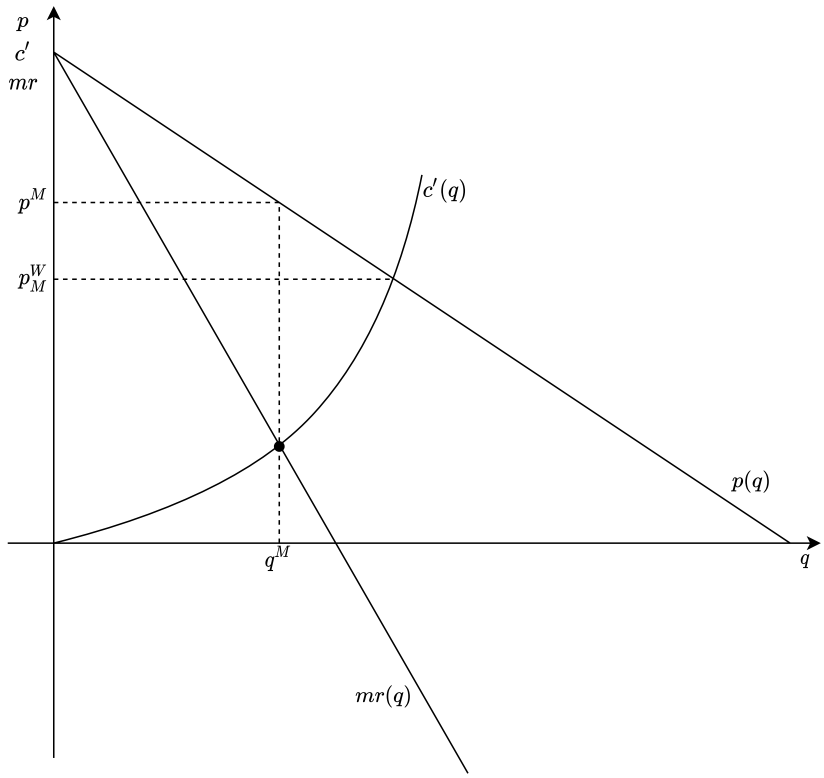


**Figure B.1.** The monopolist's profit maximization problem without price caps. The profit is maximized by the quantity at which the marginal revenue and the marginal cost intersect. The welfare is maximized at the quantity at which the inverse demand curve and the marginal cost curve intersect.

Figure B.1 illustrates the monopolist's maximization problem. As in the main part, I assume that the inverse demand curve is falling and strictly log-concave and that the marginal cost is strictly increasing. The monopolist maximizes its profit, $\pi = q \cdot p(q) - c(q)$. The marginal revenue is $p(q) + q \cdot p'(q)$, which includes the inframarginal loss from depressing the price when increasing the quantity. Log-concavity of the inverse demand function implies that the marginal revenue is decreasing whenever it is positive.[25]

In the absence of a price cap, the monopolist maximizes its profit at the intersection of the marginal revenue and the marginal cost curve: It produces the quantity $q^M$, which leads to a price of $p^M$. This quantity falls short of the welfare-maximizing quantity, $q_M^W$, at which the marginal willingness to pay of the market is equal to the marginal cost of the monopolist (which are equal to $p_M^W$).

A price cap, $\bar{p}$, changes the marginal revenue curve as the monopolist is relieved of its effect on the price—as long as the price cap binds. As a price-taker, the marginal revenue is equal to the price as long as the price does not change. When the quantity is so large that the price cap stops binding, the monopolist has again a price effect and the marginal revenue drops to the normal marginal revenue.

There are now three cases, depending on the level of the price cap.

25. See https://economics.stackexchange.com/questions/24833/can-marginal-revenue-be-increasing (last accessed September 27, 2022).

**First Case, $\bar{p} \geq p^M$.** The price cap stops binding before the marginal revenue intersects the marginal cost. Thus, the price cap has no effect and the monopolist produces $q^M$.

This is sketched in Figure B.2.

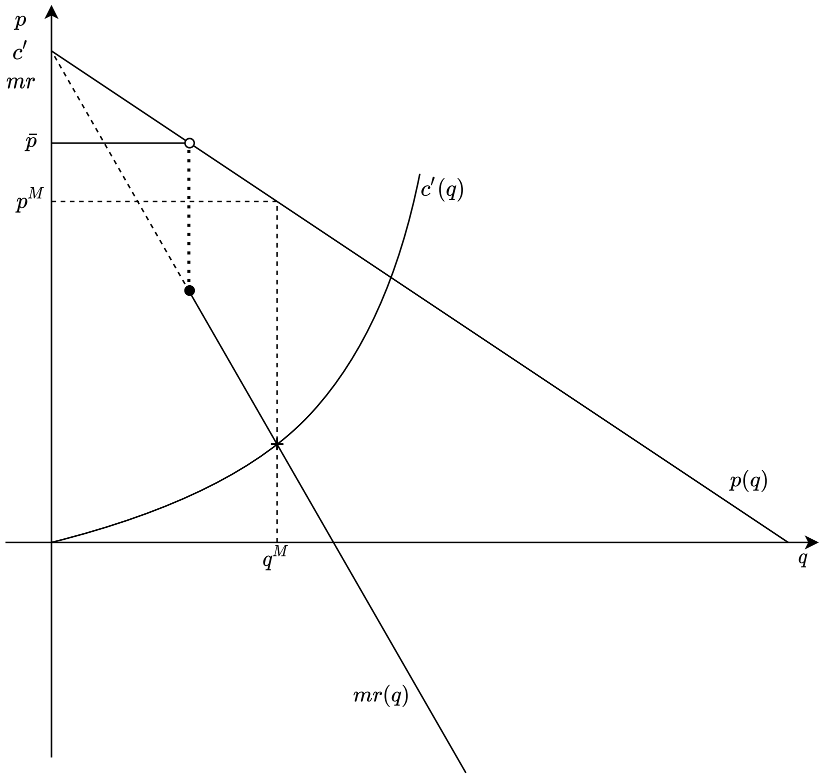


**Figure B.2.** If the price cap is above the monopoly price, the marginal revenue becomes the standard monopolist's marginal revenue before intersecting the marginal cost curve. Thus, the price cap has no effect and the profit-maximizing quantity is still the monopoly quantity.

**Second Case, $p^M > \bar{p} \geq p_M^W$.** The equilibrium quantity is determined by the inverse demand curve. The marginal revenue is constant until the price cap stops binding. At the corresponding quantity, the marginal revenue drops. Because the price cap is below the monopoly price, the marginal revenue after the drop is below the marginal cost. Thus, the quantity that the monopolist optimally produces is determined by $p(q^*) = \bar{p}$.

This is sketched in Figure B.3.

**Third Case, $p_M^W > \bar{p} > c'(0)$.** The equilibrium quantity is determined by the marginal cost. In this case, the marginal revenue intersects the marginal cost before the price cap stops binding. Thus, the quantity that the monopolist optimally produces is determined by $c'(q^*) = \bar{p}$. If the price cap is very low, the monopolist produces even less than it would without a price cap, albeit that quantity is sold at a much lower price. The lower price causes a rationing problem: At the price cap, the demand exceeds the supply; the price cannot (efficiently) allocate the good.

This is sketched in Figure B.4.

**Constant Marginal Cost.** When the monopolist has constant marginal cost, most of the above remains true. The difference is that the third case does not exist as $p_M^W = c'(0)$.

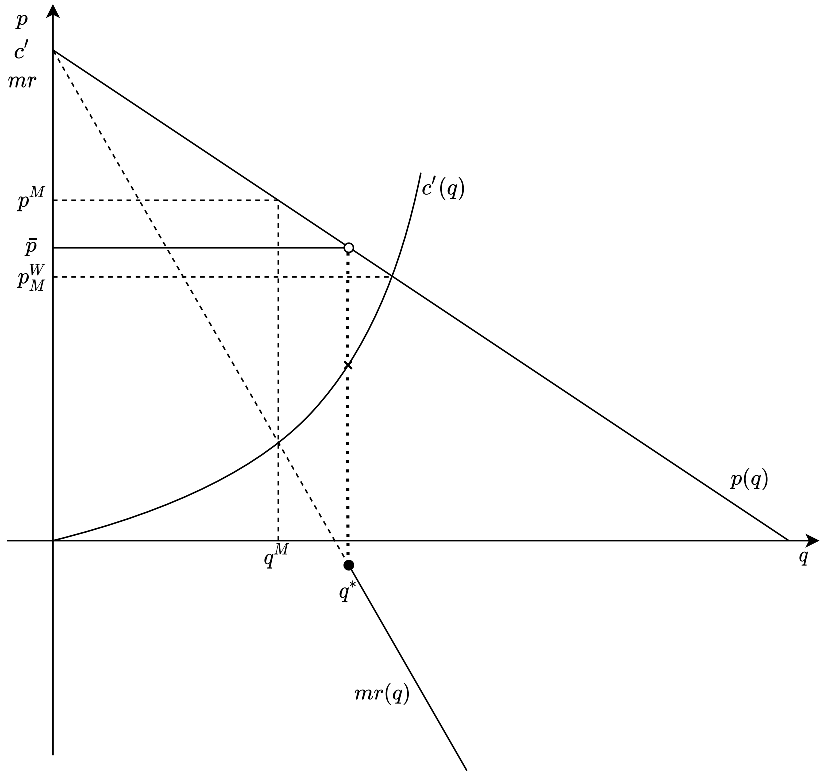


**Figure B.3.** Price caps between the monopoly price and the welfare-maximizing price increase the welfare.

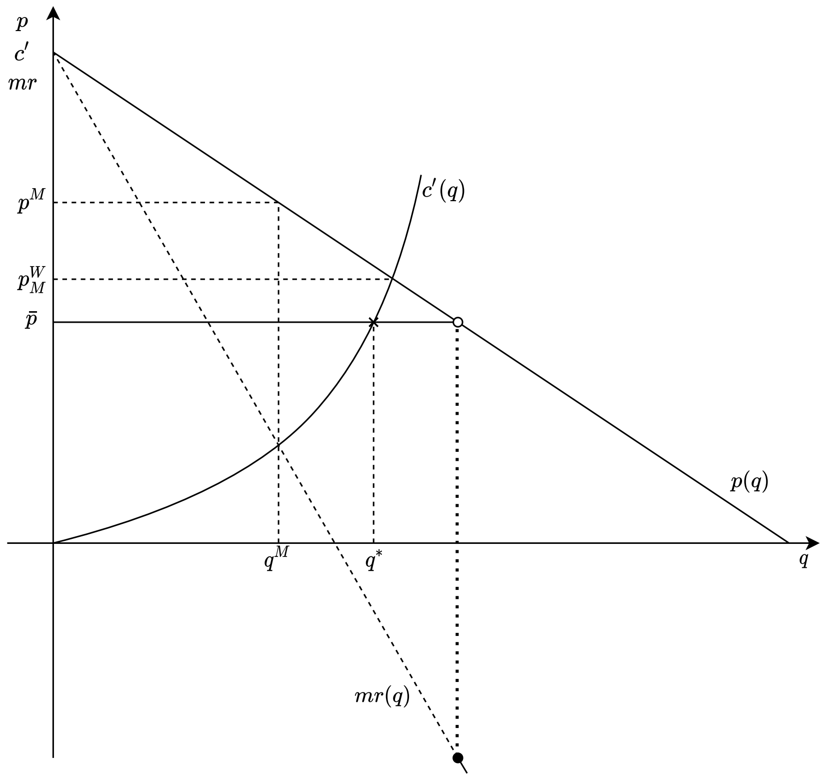


**Figure B.4.** If the price cap is below the welfare-maximizing price, the marginal revenue intersects the marginal cost in the range in which it is constant.

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